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DECISION IN BATTLE: BREAKPOINT HYPOTHESES AND ENGAGEMENT TERMINATION DATA

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Rand Corporation
Santa Monica, California

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10. ABSTRACT

The validity of breakpoint hypotheses is of interest to the Air Force because such hypotheses are imbedded in several models presently used to evaluate weapon systems in terms of the effect of air-delivered munitions on the course of a land combat engagement. This report investigates a popular assumption regarding the relationship of casualties to the decision to terminate a battle--the assumption that a military force gives up the battle when its personnel casualty fraction reaches a certain level, which may be either a fixed quantity or one determined on a probabilistic basis. Theoretical implications of a basic breakpoint hypothesis are developed, and these are quantitatively compared with casualty-fraction distribution data from various investigations of land combat. Tentative observations are offered regarding future attempts to resolve the problem of decision in battle.

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PREFACE

This study was undertaken to examine, through the use of historical data, the validity of breakpoint hypotheses as explanations of the outcomes of land combat battles. The validity of breakpoint hypotheses is of interest to the Air Force because such hypotheses are imbedded in several models being employed to evaluate weapons systems in terms of the effect of air-delivered munitions on the course of a land combat engagement.

The work reported here is a part of two larger Rand studies, the Forward Air Strike Evaluation (FAST-VAL) project and a broad study of close air support. In both of these, it was desirable to have some way of relating the outcome of a battle to the effects of personnel casualties inflicted by air. This report presents one aspect of the exploration of the range of validity of the general breakpoint hypothesis. The scope of this report is limited to exploring a popular form of assumption regarding the relationship of casualties to the decision to terminate a battle--the assumption that a military force gives up the battle when its personnel casualty fraction reaches a certain level, which may be either a fixed quantity or one determined on a probabilistic basis. Assumptions of this type are commonly used to simplify the problem of deciding when and how to terminate simulated battle engagements in war games, field maneuvers, and computer simulations. The object of the present investigation is to determine the extent to which such a procedure is justified by confronting it with available data on historical battle engagements.

This report will be of interest to persons concerned with war games or similar efforts involving the relation of the outcome of a tactical engagement to the personnel casualties incurred by the contending forces.

SUMMARY

The purpose of this report is to address the validity of a breakpoint-type hypothesis for determining the terminal status of a land battle. The primary version of the breakpoint hypothesis used is a moderate simplification of the ones frequently used to determine when and how to terminate simulated combat for various types of combat models, such as those used in war games, computer simulations, and the like. The basic breakpoint hypothesis used is as follows:

1. Each side selects independently a breakpoint from a distribution of such breakpoints and gives up the battle when its casualty fraction reaches its breakpoint.
2. These breakpoint distribution curves are generally applicable.
3. The casualty fractions of the forces are deterministically and monotonically related to each other.

Some of the major theoretical implications of this breakpoint hypothesis are developed, and these are quantitatively compared against casualty-fraction distribution data from various investigations of land combat. Some alternative versions of the basic breakpoint hypothesis are outlined and tentatively discussed in terms of the same data, to see what leads they may provide to a more satisfactory theory of the battle termination process.

The principal finding is that the breakpoint hypothesis yields theoretical implications that are at variance with the available battle termination data in several essential respects. Some tentative observations and remarks are offered regarding possible directions for future attempts to resolve the problem of decision in battle. However, the task of devising a theory that satisfactorily accounts for the available data is not within the scope of this report. Until a better theoretical explanation of the battle termination process becomes available, the soundness of models of combat such as war games and computer simulations that make essential use of breakpoint hypotheses is suspect.

ACKNOWLEDGMENTS

The author is grateful to Louis Wegner, Marvin Schaffer, and Peter Gluckman of The Rand Corporation for their careful reading and helpful comments on preliminary drafts of this report. In addition, a special debt is owed to William A. Schmieman who provided the IBM punched card deck used to obtain most of the empirical casualty-fraction distributions presented. Dr. Schmieman had used these data in preparing his Doctoral Thesis at the Georgia Institute of Technology. The card deck provided by Dr. Schmieman is his version of one originally prepared at the Research Analysis Corporation under the direction of Daniel Willard and used by Dr. Willard in preparing a paper on *Lanchester As Force in History*.

LIST OF SYMBOLS AND TECHNICAL TERMS

A = rate of defender force attrition per unit attacker troop

$C_z \equiv C_z(T)$ = total casualties suffered by side z during the battle ($C_z = z_0 - z$)

$C_z(t)$ = casualties sustained by side z as of time t into the battle; $C_z(t) = z_0 - z(t)$

D = rate of attacker force attrition per unit defender troop

$D_x(v|W_y)$ = conditional distribution of L_x, given W_y

$D_y(u|W_x)$ = conditional distribution of L_y, given W_x

$F_z(u)$ = break curve for side z, given the probability that the side's breakpoint threshold, L_z, will not exceed u; $Pr\{L_z < u\}$

$f_z \equiv f_z(T)$ = casualty fraction sustained by side z in the battle

$f_z(t)$ = casualty fraction for side z as of time t into the battle; ($f_z(t) = C_z(t)/z_0$)

L_z = preselected (breakpoint) casualty-fraction level which, if met or exceeded, results in side z's losing the battle

T = duration of the battle

W_z = the event that side z wins

x = general symbol for the attacker

y = general symbol for the defender

z = general symbol denoting a value of either x or y, depending on context

$z \equiv z(T)$ = surviving troop strength of side z at the end of the battle

z_o = initial troop strength of side z at the start of the battle

$z(t)$ = surviving troop strength of side z as of time t into the battle

$\Delta_{zz'}(u) = P(f_z < u|W_z')$

$\delta(x, a)$ = Dirac δ-function of x with spike at a so that

$$\sigma(x, a) = \int_{-\infty}^{x} \delta(t, a)\, dt$$

$$\sigma(x, a) = \begin{cases} 0, & x < a \\ \frac{1}{2}, & x = a \\ 1, & x > a \end{cases}$$

φ = a strictly increasing monotonic function relating $f_x(t)$ to $f_y(t)$ via the formula $f_x(t) = \varphi[f_y(t)]$

$\Psi(u)$ = Min $[\varphi(u), 1] = \varphi(u)\sigma(1, \varphi(u)) + \sigma(\varphi(u), 1)$

dual = the result of applying the usual transposition to a formula, expression, etc.

"usual transposition" = $x \to y$, $y \to x$, $\Psi^{-1} \to \Psi$, $\Psi \to \Psi^{-1}$

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I. INTRODUCTION

Consider two opposing forces engaged in a land battle. As the engagement continues, both sides will suffer casualties. Eventually, the battle will end. At the termination of the engagement, the situation may be one of the following:

- o One side has, for all practical purposes, been annihilated, leaving its opponent in control of the battlefield.
- o One side surrenders and submits to the will of its opponent, who thereby acquires control of the battlefield.
- o Neither side has surrendered or been annihilated, but one of them has disengaged and either has withdrawn or is in the process of withdrawing from the area, leaving its opponent rather clearly in control of the battlefield.
- o Neither side has surrendered or been annihilated, but both sides have disengaged their forces, and both sides either have withdrawn or are in the process of withdrawing their forces from the area. The withdrawal is mutual, and it is impossible, or at any rate a very difficult and controversial matter, to assert that either side has practically exclusive control of the battlefield.

This list of possibilities excludes a situation that occasionally occurs, in which both sides have disengaged their forces, but neither side appears ready to leave the field. Sporadic skirmishes may be taking place along the line of demarcation. (Typically, this sort of situation occurs when a defensive force is reluctant to leave a strong defensive position in the presence of a relatively stronger enemy who considers that an immediate assault would not be worth the probable losses.) These conditions evidently describe a kind of unstable standoff that will eventually resolve itself either into a renewal of the engagement or into one of the four kinds of termination described earlier, so we will view the standoff case as a temporary pause or lull in hostilities, rather than as a conclusion of the engagement.

Of the four terminal situations listed, the second and third, where there is a fairly clear-cut victor, seem to be the most common. Possession of the battlefield seems to be a generally accepted criterion

of victory in the battle. There are cases in which the battle loser has imposed a serious strategic cost on the tactical battlefield winner. The "Pyrrhic" victory (Battle of Asculum, 279 B.C.) is a famous example of a tactical victory obtained at a heavy strategic loss. Annihilation, except in circumstances where retreat is impossible (as may occur, for example, in sieges or in island campaigns), is quite rare. Even where retreat is out of the question, a defender whose position is deteriorating will normally surrender rather than fight to the last man. Mutual withdrawal, with its inconclusive outcome, although more frequent than annihilation, is still a relatively rare occurrence. In general, a weakening side will prefer to withdraw and abandon the field rather than surrender to its opponent, and (if withdrawal is not feasible) will usually prefer to surrender at some casualty level short of 100 percent total annihilation.

A so-called "break curve" is a device sometimes used to model the inclination of a weakening force to discontinue the engagement by acknowledging defeat and either withdrawing (if it can) or surrendering. It is a curve that purports to show the probability that a force will discontinue the engagement as a function of the casualty fraction that it has sustained. (Figure 1 shows a hypothetical break curve.) A break curve is often used in combat models as follows. At or before the beginning of a simulated engagement, a sample casualty-fraction value for each side is drawn from the distribution of such values defined by an appropriate break curve. The values so selected are called the "breakpoints" for the two sides. Then, as the engagement progresses, both sides are considered to be engaged in a contest for dominance until one of them accumulates enough casualties to equal or exceed its preselected breakpoint. At that point, the side

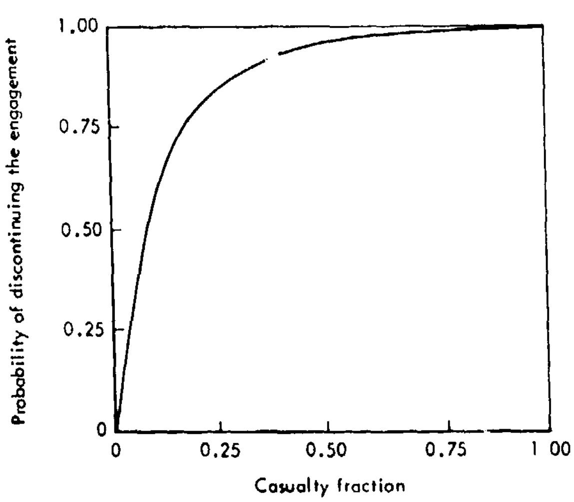


Fig. 1—Hypothetical break curve

whose preselected breakpoint has been reached is said to "break," meaning that it is presumed to discontinue or "break off" its attempts to dominate the opposing side. Thus, the side that breaks is considered by the rules of this particular model to lose the battle.

Break curves of the sort just described are presented in Ref. 1 (paragraph 15, Appendix IV). Reference 2 gives an example of their application to a particular model. Frequently, application of the break curves is simplified by assuming that breaks occur deterministically. The break-curve approach described above can be adjusted to this case by taking the break curve to be a step function with a vertical rise of unity at the deterministic breakpoint, as indicated in Fig. 2. This special type of break curve will be called a deterministic break curve. Perhaps the most common type of break curve proposed is of the deterministic type. For example, deterministic break curves have been used to determine the outcome of simulated battles in the Rand FAST-VAL model,[3] have been used by the Research Analysis Corporation in a series of small-unit simulations, and were employed by the Center for Naval Analyses to determine the subordinate unit outcomes occurring in some large-scale simulations. Other examples could be cited.

Objections to the validity of deterministic break curves as descriptors of combat behavior have been voiced from time to time. For example, according to Clark,[4] "The statement that a unit can be considered no longer combat effective when it has suffered a specific casualty percentage is a gross oversimplification not supported by combat data." The collection of casualty data included in Appendix F confirms this conclusion. Clark showed that a deterministic-type break curve is not generally applicable to the observed behavior of combat units but did not analyze the

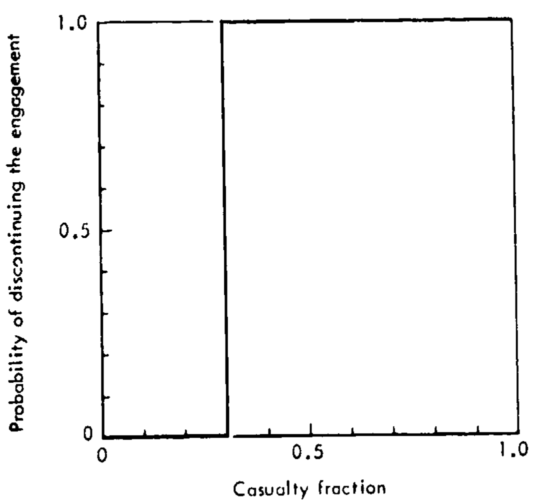


Fig.2—A deterministic break curve

validity of the more general type of break curve illustrated in Fig. 1. At present, the validity of the more general type of break curve seems to be a controversial matter. On one hand, some analysts have proposed their use for war gaming, maneuver control, and similar purposes, as noted earlier. Other analysts have designed simulations using the simpler and more specialized deterministic break curves, despite Clark's objections to their merit, and so by implication have embraced the basic philosophy that unit behavior is representable by *some* type of break curve.

On the other hand, some analysts have grave misgivings about the validity of break curves--even while they may, on occasion, use them for lack of anything better. Some of the objections raised against the use of break curves are discussed below. Most of them can be characterized as suggesting that some other factor or factors than simply the current casualty level of a force influence the break behavior of the force. Frequently these other factors are proposed as considerations supplementary to, rather than as replacements for, the casualty-level criteria. This suggests that the casualty level is often thought of as a sort of "core" consideration that may be modified in particular situations by some of these additional considerations.

For example, it is sometimes suggested that the casualty rate, as well as the casualty level, influences the behavior of a force. Other considerations include the level of training and battle experience of the troops, the influence of inclement weather or other unusual environmental stress, the importance of the mission, troop morale, the quality of leadership, the degree of knowledge and intelligence of the enemy's situation and intentions, the perceived vigor of the enemy opposition, the scale of friendly fire support and troop reinforcement, the logistical supply situation, and the availability of good communications with other friendly units. Many of the considerations that impinge on the intuitive plausibility of the break-curve approach are carefully discussed in Ref. 4. We do not intend to pursue the extent to which the break-curve model's "face validity" is affected by these plausibility arguments, since we will confront our model with empirical data in order to determine its validity.

However, there is one further objection that has been raised against the break-curve approach that needs to be discussed in somewhat more detail. This is the observation that each side in an actual battle surely considers the progress of the battle and continually assesses its own situation relative to that of its opponent, rather than being governed solely by its own condition. In this view, each side conducts itself according to the results of a dynamic decision process lasting throughout the battle rather than preselecting a specific breakpoint, as is done in the conventional application of break curves to war games, simulations, and field maneuvers. That the objection is not always relevant can be shown by the discussion in Appendix C, where it is shown how some types of continuous decision process can be subsumed under the break-curve paradigm without losing any generality. The key assumption in such derivations is the supposition that each side, while it may decide continually whether to continue the engagement or not, bases the decision solely on its own current casualty fraction. Similar derivations of break curves from dynamic decision processes have been given in Refs. 5, 6, and 7. In none of these derivations is the possibility that one side's breakpoint may depend on the casualty level of its foe explicitly considered. Thus, it seems that in order for the objection raised earlier (that break curves fail to reflect the dynamic decision processes actually taking place in combat) to retain its validity it must also be supposed as a minimum that one side's breakpoint distribution depends on the other side's casualty level.

In addition to the conceptual issues discussed above, there are several practical problems in assessing the validity of breakpoint assumptions. These stem from the kind of empirical evidence that is more-or-less readily available for comparisons with the model. First, the recoverable data are essentially limited to estimates of the attacker and defender initial troop strength, of the total losses* on each side, and (occasionally) of the temporal duration of the battle, together

*Not necessarily only those inflicted prior to reaching a breakpoint. In some cases, the historically reported casualties may have occurred *after* the break. For example, routs sometimes degenerate into massacres, and on occasion troops that have surrendered may have been slain.

with a narrative account of the action and an historical judgment either awarding the victory to one side or the other or declaring the outcome "indecisive." Second, the criteria for assessing casualties may vary among battle descriptions from very broad to highly restrictive. Third, there is often much scope for human error and/or capriciousness in selecting the forces to be included in establishing troop strength or casualties, as well as in arriving at an accurate inventory of these quantities. These problems are noted and discussed a bit further in Ref. 8, but no solution to them (short of a reexamination of the original historical records) is in evidence. These problems make enlarging the sample size a generally tedious, time-consuming, and often expensive task. Such is the nature of the basic data at our disposal.

To the above difficulties yet another must be added--namely that the attrition dynamics intervene between the break curve and the observed battle outcome and force ratio. That is, after breakpoints are established, parallel casualty assessments for each side must be made in order to determine the final outcome and casualty fractions. Consequently, it is clearly incorrect to establish a break curve by simply plotting the cumulative fraction of battles that terminated before various casualty-fraction levels were sustained. A correct analysis of the relation of observed casualty-fraction distributions and break curves is given in the next section. Later sections present some empirical battle data and discuss their relation to the model.

II. BREAKPOINT MODEL

The breakpoint model considered here is founded on the following postulates. The ensuing development requires each of the assumptions made, as well as some additional ones that will be introduced as we go along.

BREAKPOINT HYPOTHESIS

Hypothesis A. Termination of a battle can be considered as governed by the following mechanism, or one that gives the same results: Prior to the battle, each side independently and at random selects a casualty-fraction value (breakpoint) from some distribution of casualty fractions. When either side experiences a casualty fraction equal to the preselected breakpoint, the battle terminates with a loss to the side that "broke."*

Hypothesis B. The breakpoint distributions (break curves) mentioned above are generally applicable. That is, they are the same for all battles, irrespective of the size of forces involved or when, where, by whom, or with what the battle was fought.

Hypotheses A and B are introduced because that is the way break curves are used in many war games and combat simulations. Hypothesis B can be tested by various groupings of empirical battle data, and also

---

*In employing the casualty-fraction value as the key parameter value, there is a tacit assumption that the battle is fought to its conclusion with the forces on hand at the start, since this provides a well-defined base for establishing the casualty fraction. If reinforcements occur during the battle, then it is necessary to have some further rules about how to determine the casualty fraction. For example, Clark[4] computes distinct casualty-fraction values two ways: (1) cumulative casualties from start of engagement per troop at the start, and (2) cumulative casualties less cumulative replacements per troop at the start. In other contexts, reinforcements are often modeled in one of two extremes, i.e., either they are assumed to have a negligible impact on the situation and ignored (perhaps with some rationalization to the effect that they arrived too late to affect the outcome), or they are lumped with the initial forces and so are counted as being fully effective throughout the battle. In this report, we shall take the initial forces given in the references consulted as the base for determining casualty fractions.

makes explicit an assumption that is often overlooked. Hypothesis B is, to a large extent, provisional, in that we may modify it if the empirical data warrant it. It is certainly a rather strong and perhaps controversial assumption, once it is clearly stated. However, it is hoped that it may be testable, whereas the opposite tack of assuming that every battle fought has its own special break curves which depend on the unique circumstances surrounding the particular battle is not likely to lead to a theory that can be compared with such data as are available.

While data from which accurate curves may be drawn are hard to come by, there is no other reason for restricting the method to a single break curve. In principle, the appropriate break curve could be made to depend on any condition that could be known at the time the break curve is sampled, such as whether the force is initially attacking or defending, its state of training, experience, morale, physical weariness, etc. We will not pursue this possibility here. The approach adopted is in keeping with the spirit of Richardson's Principle to the effect that "formulae are not to be complicated without evidence." (See Ref. 9, p. xliv.)

Some notation needs to be introduced at this point (see also the List of Symbols and Technical Terms). Let $f_x(t)$ and $f_y(t)$ be the fraction of casualties for side x (attacker) and side y (defender) as of time t after the start of the battle. Let L_x and L_y be the breakpoints or casualty-fraction threshold values for the attacker (side x) and defender (side y), respectively. Let f_x and f_y be the fraction of casualties sustained by the attacker and the defender during the whole course of the engagement.

By virtue of the breakpoint hypothesis, L_x and L_y are random variables with appropriate distributions. Either f_x or f_y is equal to its corresponding breakpoint, while the other is less. Thus, we have either $f_x < L_x$ and $f_y = L_y$ (in which case the attacker wins) or $f_x = L_x$ and $f_y < L_y$ (in which case the defender wins). In either case, both $f_x(t) < L_x$ and $f_y(t) < L_y$ hold for all times t from onset of the battle to its conclusion, i.e., for $0 \le t \le T$.

At this point, we introduce Hypothesis C.

<u>Hypothesis C</u>. The losses, and hence equivalently the casualty fractions, of the forces are deterministically and monotonically related to each other. That is, there is a monotonically increasing function, $\varphi(\cdot)$, such that

$$f_x(t) = \varphi[f_y(t)], \qquad 0 \le t \le T.$$

It would be of interest to consider the effect of assuming nondeterministic and/or nonmonotonic relationships between the two casualty fractions, although such an investigation is not within the scope of this analysis. The assumption made here is a generalization of that made by Weiss, who assumes that the casualty fractions are proportional to each other (see Ref. 7, p. 776), i.e., that there is an "exchange ratio," R, such that*

$$f_x(t) = Rf_y(t).$$

This is equivalent (provided, of course, that R > 0) to the special case of $\varphi(u) = Ru$. At a later point in the argument, we will find it useful to introduce particular forms of the function φ. The real reason for assuming φ to be strictly monotonic is to assure that it will have a uniquely definable inverse, φ^{-1}, whose role is made clear by ensuing developments.

<u>DERIVATION OF FORMULAE FROM THE BREAKPOINT HYPOTHESIS</u>

If the attacker is to win, then we must have

$$f_x(t) < L_x, \qquad 0 \le t \le T$$

---

*The details of Weiss's subsequent development diverge from ours in that he introduces a model of break behavior in terms of a continual, but mutually independent evaluation of current status by each side. However, as was noted earlier, the approach presented here applies to this case also, once the break curves for each side have been derived from the dynamic model of each side's decision behavior (see Appendix C).

and

$$f_y = f_y(T) = L_y.$$

In particular, if the attacker wins, we must have

$$L_x > f_x = \varphi(f_y) = \varphi(L_y).$$

Conversely, if

$$\varphi(L_y) < L_x,$$

then, since

$$f_y(t) \le L_y, \qquad 0 \le t \le T,$$

it follows, by the monotonicity of φ, that

$$\varphi(f_y(t)) \le \varphi(L_y) < L_x, \qquad 0 \le t \le T,$$

and then using $f_x(t) = \varphi(f_y(t))$,

$$f_x(t) \le \varphi(L_y) < L_x, \qquad 0 \le t \le T,$$

and the attacker wins. Thus, the attacker wins if, and only if,

$$\varphi(L_y) < L_x.$$

Since we intend that battle outcomes be "almost always"* well-defined by our model, we could assign victory to the defender when $\varphi(L_y) = L_x$, or we could see to it that this equality has zero probability of occurrence. For some purposes, it may be convenient to adopt the convention that the battle is a toss-up when $\varphi(L_y) = L_x$, and to

*That is, except (possibly) for an event with zero probability of occurrence.

award victory with equal probability to both sides. In any case, we arrange things so that

$$P(W_x) = 1 - P(W_y),$$

where $P(W_z)$ is the probability of a win for side z, z = x or y.

Let

$$F_z(u) = P\{L_z \leq u\}$$

be the break curve for side z. Now, $F_z(0) \neq 0$ would imply that there is some positive probability that side z would break while its casualty fraction was zero, which may physically be interpreted as a refusal to engage in battle on the part of side z. Since we wish to consider only cases where the battle has been joined, we take

$$F_z(0) = 0.$$

Also, $F_z(1) \neq 1$ would imply that side z might not break even when its casualty fraction was unity. This seems to be intuitively unreasonable, and so we assume that*

$$F_z(1) = 1.$$

We now wish to express $P(W_y)$ in terms of the F_z's. To do this we begin by noting that the preceding discussion of the conditions under which the defender wins yields the following relationship

$$P(W_y) = P\{L_x \leq \varphi(L_y),\ 0 \leq L_y \leq 1\},$$

---

*Strictly speaking, since we adopt the convention that distribution functions are defined by their limits from the right, this should read $F_z(1 + 0) = 1$, and similarly the previous assumption is more exactly expressed by $F_z(0 + 0) = 0$. In some cases these technicalities are important; in others, they are not.

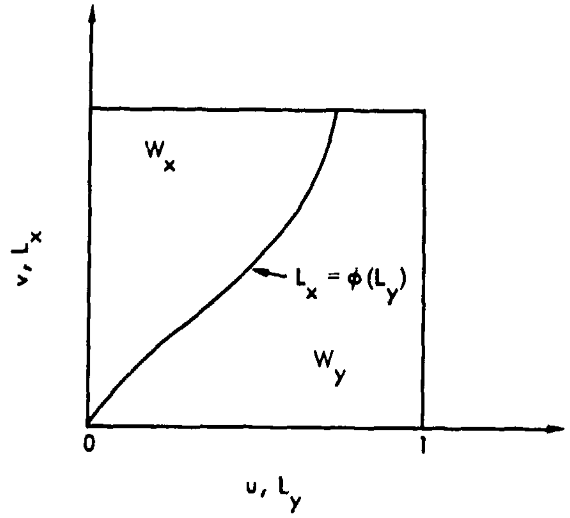


Fig.3—Relationship between L_x and L_y

since $L_x > \varphi(L_y)$ if, and only if, the attacker wins. To calculate $P(W_y)$, we consider the schematic diagram shown as Fig. 3. The set of points $(L_y, L_x) = (u, v)$ for which the attacker wins is marked by W_x, and similarly for defender wins by W_y. The joint density of (L_y, L_x) is, by Hypothesis A, given by

$$dF_x(v)\ dF_y(u),$$

and so,

$$P(W_y) = \int_{u=0}^{1} \int_{v=0}^{\Psi(u)} dF_x(v)\ dF_y(u) \tag{1}$$

$$= \int_{0}^{1} F_x(\Psi(u))\ dF_y(u),$$

where we have truncated $\varphi(u)$ by setting*

$$\Psi(u) = \text{Min}\ [\varphi(u), 1].$$

Similarly,

$$P(W_x) = \int_{v=0}^{1} \int_{u=0}^{\Psi^{-1}(v)} dF_x(v)\ dF_y(u),$$

---

*$\varphi(u)$ is assumed to be monotonically increasing and defined for all $0 \leq u \leq 1 + 0$.

which becomes

$$P(W_x) = \int_0^1 F_y(\Psi^{-1}(v))\ dF_x(v). \tag{2}$$

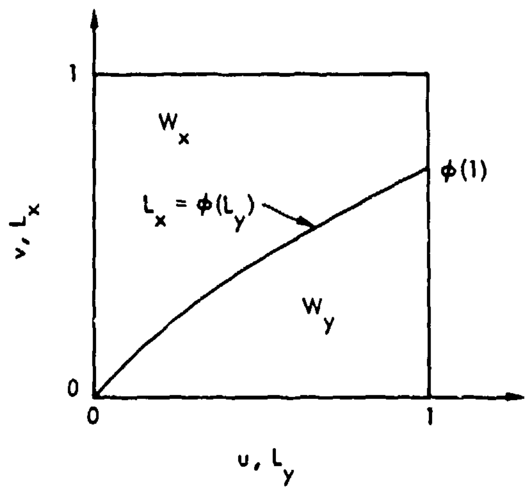


Fig. 4—Another possible relation between L_x and L_y

If $\varphi(1) < 1$, as illustrated in Fig. 4, then we define $\Psi^{-1}(v) = 1$ for $\varphi(1) \le v \le 1$. This manner of defining the inverse function preserves the correctness of the formulae just given.

By integrating in the reverse order with respect to the variables u and v, we obtain formulae equivalent to those that would result from an integration by parts, thus,

$$P(W_y) = \int_{v=0}^{1} \int_{\Psi^{-1}(v)}^{1} dF_y(u)\ dF_x(v) = \int_0^1 \left[1 - F_y(\Psi^{-1}(v))\right] dF_x(v) \tag{3}$$

and

$$P(W_x) = \int_{u=0}^{1} \int_{\Psi(u)}^{1} dF_y(u)\ dF_x(v) = \int_0^1 \left[1 - F_x(\Psi(u))\right] dF_y(u). \tag{4}$$

From Figs. 3 and 4, we see that the conditional joint density of (L_y, L_x),

given W_x, is

$$\frac{dF_y(u)\ dF_x(v)}{P(W_x)}, \qquad \text{for } (u,\ v) \in W_x.$$

So the conditional density of L_y, given W_x, is

$$dD_y(u|W_x) = \int_{v=\Psi(u)}^{1} \frac{dF_y(u)\ dF_x(v)}{P(W_x)}$$

$$= \frac{[1 - F_x(\Psi(u))]\ dF_y(u)}{P(W_x)} . \qquad (5)$$

Integration of this expression with respect to u from u = 0 to u = 1 and comparing the result with Eq. (4) shows that it represents a proper probability density.

We now find the conditional distributions of casualty fractions on each side when the attacker wins. We begin by recalling that when x wins, $L_y = f_y$. But we have just found the density of L_y when x wins. Hence,

$$P(f_y < q|W_x) = \int_0^q d\ D_y(u|W_x)$$

$$= D_y(q|W_x). \qquad (6)$$

Since $f_x = \Psi(f_y)$, the conditional distribution of the attacker's casualty fraction when the attacker wins is

$$P(f_x < s|W_x) = P(\Psi(f_y) < s|W_x)$$

$$= D_y(\Psi^{-1}(s)|W_x). \qquad (7)$$

In similar fashion we find the conditional density of L_x given W_y as

$$d\,D_x(v|W_y) = \int_{u=\Psi^{-1}(v)}^{1} \frac{dF_y(u)\ dF_x(v)}{P(W_y)}$$

$$= \frac{[1 - F_y(\Psi^{-1}(v))]\ dF_x(v)}{P(W_y)}\ . \qquad (8)$$

Since $L_x = f_x$ whenever y wins, the conditional distribution of the attacker's casualty fraction when the defender wins is just

$$P(f_x < s|W_y) = D_x(s|W_y). \qquad (9)$$

Since $f_x = \Psi(f_y)$, the conditional distribution of the defender's casualty fraction when he wins is

$$P(f_y < q|W_y) = P(f_x < \Psi(q)|W_y)$$

$$= D_x(\Psi(q)|W_y). \qquad (10)$$

These are the basic relations with which we shall work throughout the rest of the report. A collection of formulae for convenient reference is given in Appendix D, and some particular cases are worked out in Appendix E. It may be helpful to point out a "duality" property possessed by these relations. For example, Eq. (10) can be obtained from Eq. (7) by substituting x for y and y for x throughout, and substituting Ψ for Ψ^{-1}. This series of substitutions ($x \to y$, $y \to x$, $\Psi \to \Psi^{-1}$, $\Psi^{-1} \to \Psi$) will be called the "usual transposition." Formulae obtainable from each other by invoking the usual transposition will be called duals of each other. Thus, we have just shown that Eq. (10) is dual to Eq. (7), and vice versa. Equations (1) and (3) are duals; Eqs. (2) and (4) are duals; and Eqs. (5), (6), and (7) are dual to Eqs. (8), (9), and (10), respectively. Clearly, any relation correctly derived from these equations

has a dual relation obtainable from it by the usual transposition. The derivation of this dual relation is obtainable by performing the usual transposition on each step of the original derivation. This property of duality will be exploited in the following material to reduce the amount of algebraic manipulation required. The usual transposition can be applied to diagrams, expressions, etc., as well as to equations; we will make use of duality in such cases as well.

THE USE OF OBSERVED CASUALTY-FRACTION DISTRIBUTIONS TO TEST THE BREAKPOINT HYPOTHESIS

In the preceding paragraphs we have set down in explicit terms the breakpoint hypothesis (Hypotheses A, B, and C) and have shown how to derive from these hypotheses formulae that purport to describe empirical casualty-fraction distributions. In carrying out this derivation, we have been careful to maintain the essential distinction between a break curve (which is a distribution of L_z breakpoint values) and a casualty-fraction distribution (which is a distribution of f_z values). In this paragraph we show how observed casualty-fraction distributions can be used to test the breakpoint hypothesis.

We begin by recalling relations (6) and (7), which are

$$P(f_y < q|W_x) = D_y(q|W_x) = \Delta_{yx}(q) \tag{6}$$

and

$$P(f_x < s|W_x) = D_y(\Psi^{-1}(s)|W_x) = \Delta_{xx}(s), \tag{7}$$

where the Δ_{yx} and Δ_{xx} notation is introduced as an abbreviation. Combining relations (6) and (7) yields

$$\Delta_{xx}(s) = P(f_x < s|W_x) = P(f_y < \Psi^{-1}(s)|W_x) = \Delta_{yx}(\Psi^{-1}(s)), \tag{11}$$

with a dual result obtainable by the usual transposition $x \rightarrow y$, $y \rightarrow x$, $\Psi^{-1} \rightarrow \Psi$, $\Psi \rightarrow \Psi^{-1}$.

Now suppose that we had a graphical plot of the observed casualty

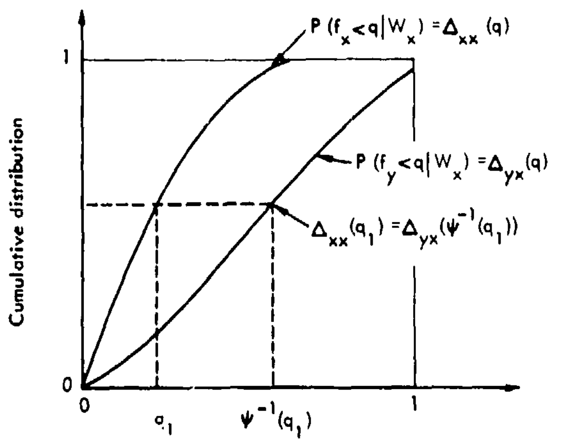


Fig.5 — Hypothetical casualty-fraction distribution in battles won by the attacker

fractions for a collection of battles that were won by the attacker. A hypothetical plot is shown in Fig. 5-- and there will be dual plot whose labels are obtainable from Fig. 5 by the usual transposition, although the curves may, of course, be differently shaped on the dual. We have indicated by the dashed lines how, using Eq. (11), the value of $\Psi^{-1}(q_1)$ can be graphically read off this plot. An exactly analogous procedure applied to the dual plot will yield the value of $\Psi(q_1)$. By repeating the process for several values of q_1 and interpolating, it is thus possible to determine suitable approximations to the functions Ψ and Ψ^{-1}.

Now, Ψ is the functional relation between f_x and f_y, since from the definition of Ψ, we may write without loss of generality

$$f_x(t) = \Psi[f_y(t)].$$

Having determined Ψ and Ψ^{-1} by the graphical procedure just described, we may plot these functions on a graph and see whether or not they obey the necessary mathematical relationship between inverse functions, that is, whether or not Ψ is a reflection of Ψ^{-1} in the 45-deg line through the origin, as illustrated in Fig. 6. If Ψ and Ψ^{-1} obey the inverse functional relationship, then this would tend to support the breakpoint hypothesis If Ψ and Ψ^{-1} do not obey the necessary mathematical relationship between inverse functions, then the breakpoint hypothesis would be definitely disproven. We shall carry out just such a test in a subsequent section.

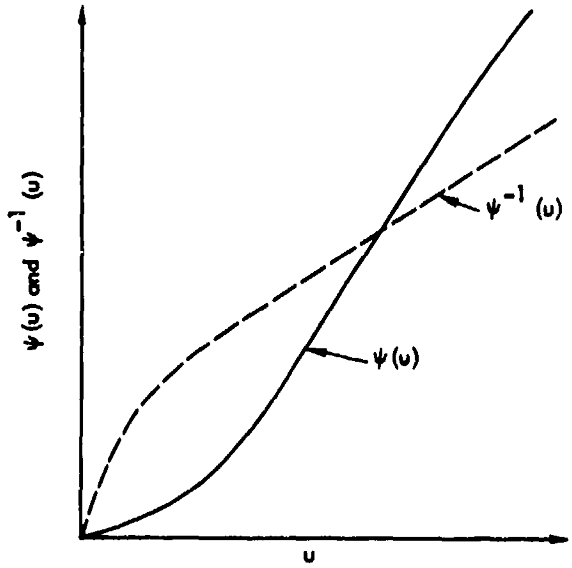


Fig.6—Inverse functional relationship

IMPORTANT LEMMAS

At this point we pause to present two lemmas that will play a prominent role in the sequel.

Lemma 1: If

$$\Delta_{xx}(u) = \Delta_{yy}(u)$$

and

$$\Delta_{yx}(u) = \Delta_{xy}(u),$$

then $\Psi = \Psi^{-1} = I$, where I is the identity function.

Proof. If $\Delta_{yx}(u) = \Delta_{xy}(u)$, then by Eq. (11) and its dual we have

$$\Delta_{xx}(\Psi(u)) = \Delta_{yx}(u) = \Delta_{xy}(u) = \Delta_{yy}(\Psi^{-1}(u)),$$

which may be written as

$$\Delta_{xx}(s) = \Delta_{yy}(\Psi^{-2}(s)).$$

But since by assumption $\Delta_{xx}(s) = \Delta_{yy}(s)$, this last implies

$$\Delta_{yy}(\Psi^{-2}(s)) = \Delta_{yy}(s).$$

By dualizing the above argument, we may also conclude that

$$\Delta_{xx}(\Psi^{2}(s)) = \Delta_{xx}(s).$$

Since the Δ's are cumulative probability distribution functions, they have definable inverses. As long as s does not lie within an interval

of constancy for both Δ_{xx} and Δ_{yy}, we conclude that

$$\Psi^2 = \Psi^{-2} = I.$$

But then we must have

$$\Psi = \Psi^{-1} = I.$$

If s does lie within an interval of constancy for both Δ_{xx} and Δ_{yy}, we may define Ψ to be I within that interval without affecting the Δ functions, and we shall adopt this convention to dispose of the ambiguity for such a case. This completes the demonstration.

<u>Lemma 2</u>: If $\Psi(s) \geq s$ for some s, then

$$\Delta_{yy}(s) \geq \Delta_{xy}(s),$$

and

$$\Delta_{xx}(s) \leq \Delta_{yx}(s).$$

Conversely, if $\Psi(s) \leq s$ for some s, then

$$\Delta_{yy}(s) \leq \Delta_{xy}(s),$$

and

$$\Delta_{xx}(s) \geq \Delta_{yx}(s).$$

<u>Proof</u>: Recall that the Δ's are distribution functions so that $v \geq u$ implies

$$\Delta_{zz'}(v) \geq \Delta_{zz'}(u).$$

Then if $\Psi(s) \geq s$,

$$\Delta_{yy}(s) = \Delta_{xy}(\Psi(s)) \geq \Delta_{xy}(s),$$

where the equality follows from the dual of Eq. (11). When the usual transposition is applied, the inequality must, of course, be reversed, since $\Psi(s) \geq s$ implies that $\Psi^{-1}(s) \leq s$. This completes the first part of the lemma. The second part is demonstrated by a similar procedure.

III. COMPARISON OF MODEL WITH DATA

PRELIMINARY COMPARISON

We first turn our attention to some of the available casualty-fraction distribution data. Later we shall use some of these data to obtain a test of the breakpoint hypothesis.

Some empirical data on casualty-fraction distributions are given in Table 1* and graphically displayed in Fig. 7. The values for $\Psi(q)$ and $\Psi^{-1}(q)$ read graphically from these figures are listed in Table 2 and plotted in Fig. 8. There is clearly a practical equality of the estimated Ψ and Ψ^{-1} functions for $0 \le q \le 0.18$, but a divergence for higher values of q. Part of this divergence may be due to the presence of 13 battles with unusually high defender casualty fractions. These battles are individually identified below:

	Defender Casualty Fraction (percent)
Alamo	97
Attu	100
Blenheim	67
Bronkhurst-Spruit ..	60
Eniwetok	100
Indus	63
Iwo Jima	100
Kwajalein North	98
Kwajalein South	86
Lesno	67
Monongahela	63
Ravenna	75
Saipan	92

Part of the divergence is also due simply to the difficulty of accurately determining the abscissa value corresponding to a given ordinate value when the slope of the curve is small, as occurs at higher values of q.

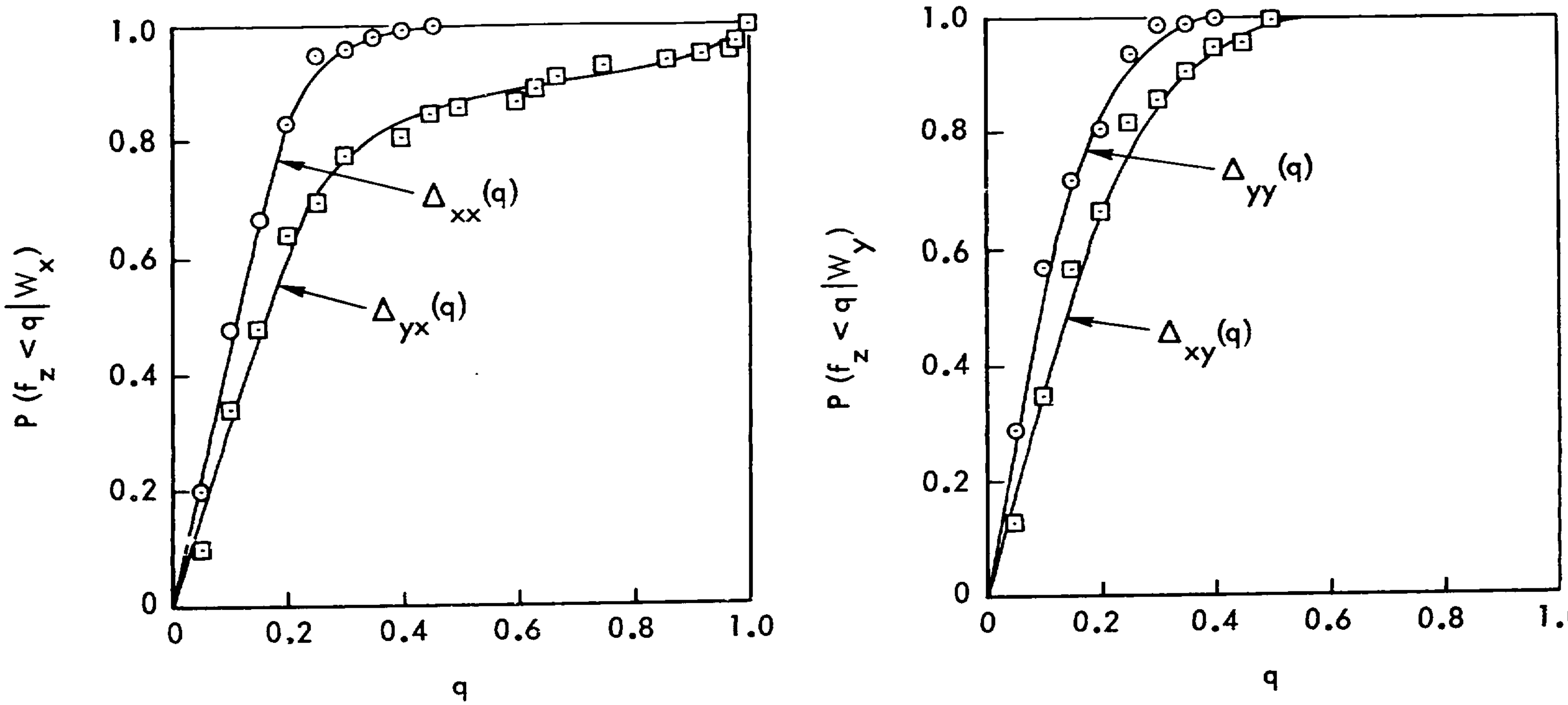
*Some of the points with high casualty-fraction values plotted in Fig. 7 are not listed in detail in Table 1, but were obtained from Refs. 8 and 10. The detailed values on which the points referred to are based are given in the battle listing above.

Table 1

EMPIRICAL CASUALTY-FRACTION DISTRIBUTIONS[a]

Range of Casualty-Fraction Values	Attacker Wins (W_x)						Defender Wins (W_y)					
	Attacker			Defender			Attacker			Defender		
	No. Battles	Cum. No. Battles	Cum. %	No. Battles	Cum. No. Battles	Cum. %	No. Battles	Cum. No. Battles	Cum. %	No. Battles	Cum. No. Battles	Cum. %
0.00–0.05	19	19	20	9	9	10	10	10	13	23	23	29
0.05–0.10	26	45	48	23	32	34	18	28	35	22	45	57
0.10–0.15	18	63	67	13	45	48	17	45	57	12	57	72
0.15–0.21	15	78	83	15	60	64	8	53	67	7	64	81
0.20–0.25	11	89	95	6	66	70	12	65	82	10	74	94
0.25–0.30	1	90	96	7	73	78	3	68	86	4	78	99
0.30–0.35	2	92	98	3	76	81	4	72	91	0	78	99
0.35–0.40	1	93	99	0	76	81	3	75	95	1	79	100
0.40–0.45	1	94	100	4	80	85	1	76	96	0	79	100
0.45–0.50	0	94	100	1	81	86	3	79	100	0	79	100
0.50–1.00	0	94	100	13	94	100	0	79	100	0	79	100

[a]Composite compiled from Table X in Refs. 8 and 10.



(a) Attacker wins

(b) Defender wins

Fig. 7 — Empirical casualty-fraction distribution (composite from Refs. 8 and 10)

Table 2

VALUES OF $\Psi(q)$ AND $\Psi^{-1}(q)$ READ GRAPHICALLY FROM FIG. 7[a]

q	$\Psi^{-1}(q)$	$\Psi(q)$
0.02	0.03	0.03
.04	.06	.06
.06	.09	.09
.08	.12	.12
.10	.15	.15
.12	.17	.18
.14	.21	.21
.16	.24	.23
.18	.29	.26
.20	.39	.29
.22	.48	.32
.24	.57	.35
.26	.66	.38
.28	.76	.41
.30	.80	.44
.32	.85	.48
.34	.90	.50
.36	.96	.53
.38	.97	..
.40	.97	..
.42	.97	..

[a]Values shown were read from the smooth curves fitted by eye to the data and shown on the figures. Values for $\Psi^{-1}(q)$ are from part (a) of Fig. 7; those for $\Psi(q)$ from part (b).

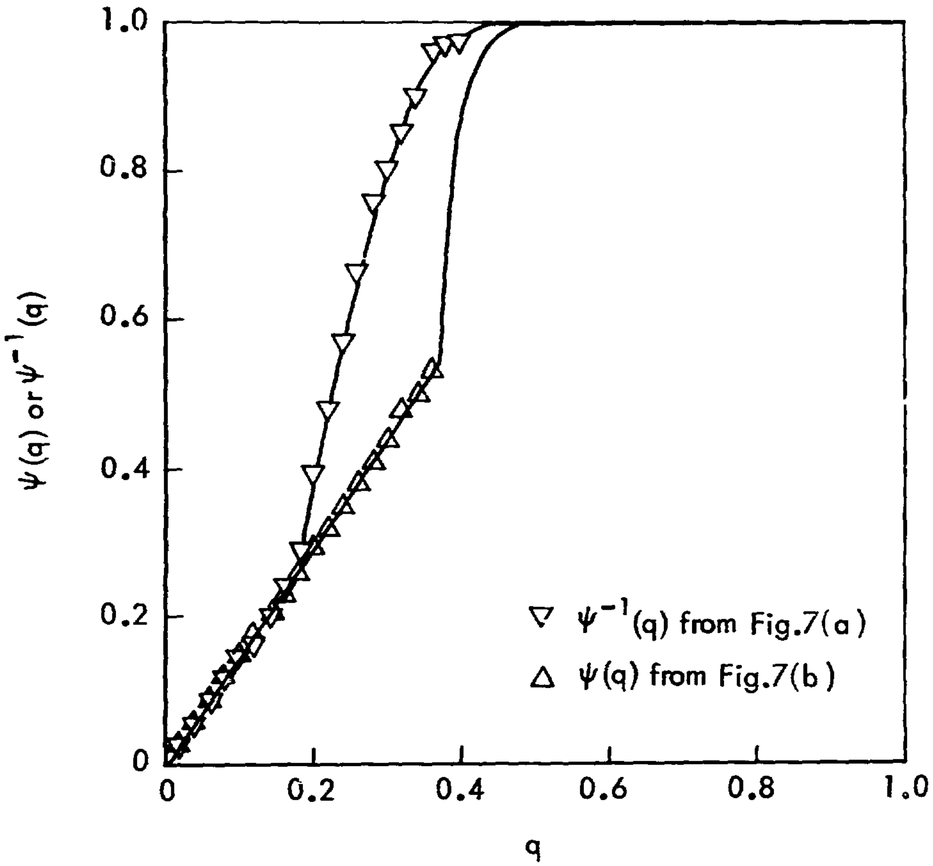


Fig.8—Values of ψ and ψ^{-1} derived from Fig.7

According to the theory developed up to this point, Ψ and Ψ^{-1} were supposed to be inverse functions. This is obviously not the case depicted in Fig. 8. Rather than being inverse functions, they are nearly equal over a significant range of argument. Even when they diverge, they most certainly do not exhibit any inverse functional relation. Accordingly, there is a serious defect in the theory so far developed.

SOME ADDITIONAL DATA AND FURTHER TESTS

The material of the preceding discussion is very damaging to the breakpoint hypothesis. In the following, we confirm and extend the results of that discussion by a second and much larger sample of data. For this purpose, it was possible to use a large sample of data extracted from Bodart's *Kriegs-Lexicon*[11] by Willard,[12] as modified by Schmieman.[13] This sample of battle data (which we shall call the Bodart data) contains 1080 battles, with casualty-fraction data for both sides in the battle, and can be used to generate casualty-fraction distributions useful for testing the breakpoint hypothesis.*

We shall actually make three distinct tests with these data, by successively selecting three distinguishable groupings of the Bodart data. The first such grouping will be the entire set of 1080 battles and so will be the same as the Bodart data sample itself. The second such grouping will be the subset of the Bodart data consisting of what Willard[12] calls the Category I battles, and includes battles described in the *Kriegs-Lexicon* as *treffen, gefecht, or schlact.* These denote "open" battles in the sense that both sides could, with about equal facility, disengage and conduct an orderly withdrawal. The third grouping consists of what Willard calls the Category II battles, and includes battles described in the *Kriegs-Lexicon* as *belagerung, einnahme, ersturmung, kapitulation,* and *uberfall.* These mainly denote "closed" battles in the

*Data from Bodart's *Kriegs-Lexicon* were also used by Smith and Donovan[14] to find the casualty-fraction distributions for the winner and loser curves shown in 8-W and 8-L in Fig. F-1 in Appendix F of this report.

sense that one of the parties in the battle is encircled or otherwise in a position from which an orderly withdrawal cannot readily be made, and whose options for maneuver are correspondingly markedly more restricted than those of his opponent. The Category I and Category II battles are nonoverlapping exhaustive subsets of the Bodart data sample and thus form a partition of it.

Casualty-fraction distributions for the entire set of Bodart data are shown in Fig. 9. Inspection of parts (a) and (b) of Fig. 9 suggests that the distribution of attacker's casualties when the attacker wins is about equal to the distribution of defender's casualties when the defender wins; and that the distribution of defender's casualties when the attacker wins is about equal to the distribution of attacker's casualties when the defender wins. These two observations may be expressed in symbols more concisely as

$$\Delta_{xx}(u) = \Delta_{yy}(u) \tag{12}$$

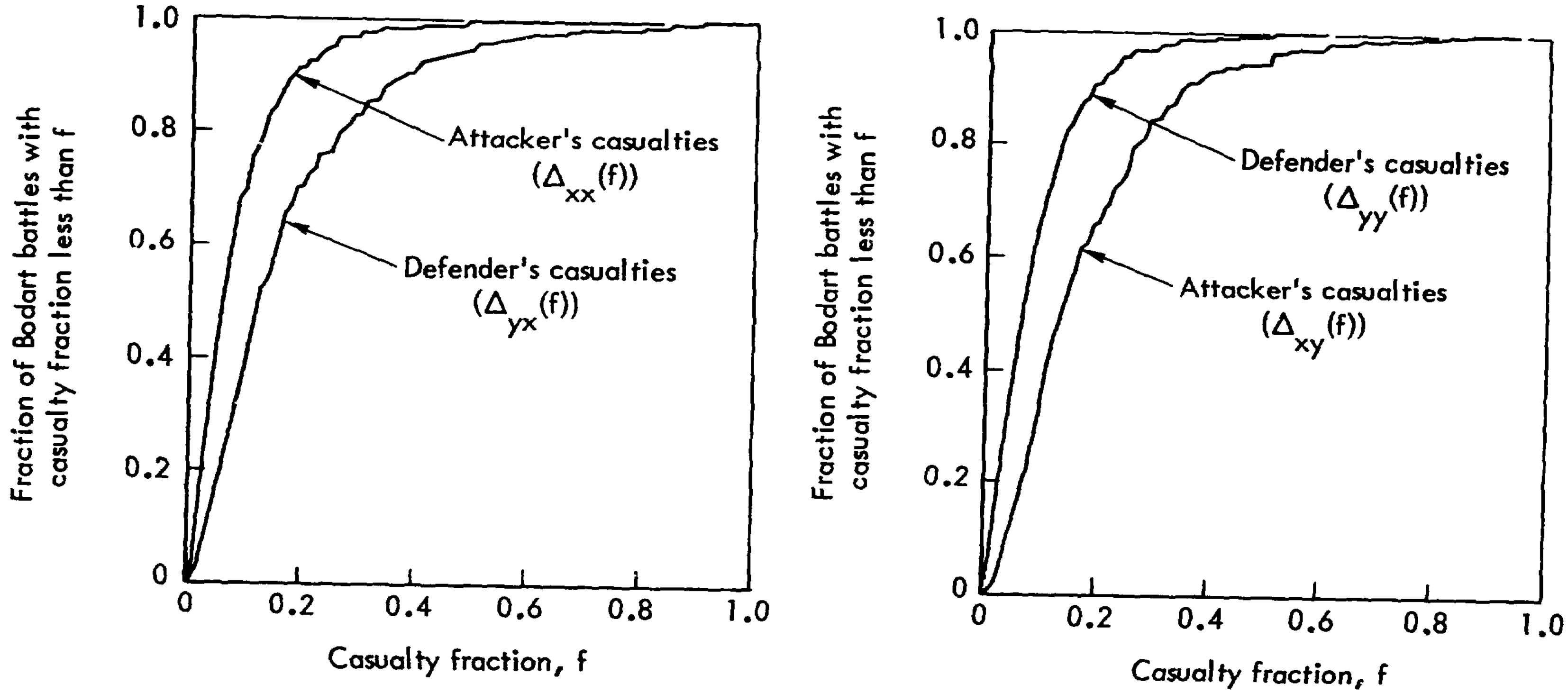
$$\Delta_{yx}(u) = \Delta_{xy}(u). \tag{13}$$

Temporarily accepting the validity of these relations, we conclude from Lemma 1 developed earlier that

$$\Psi = \Psi^{-1} = I.$$

But this conclusion is contradicted by the evidence. This may easily be seen by checking a few points on the Ψ and Ψ^{-1} curves based on the distributions of Fig. 9, which show that $\Psi(u) = \Psi^{-1}(u) = 2u$, approximately. Thus, we can show the breakpoint hypothesis to be untenable by validating relations (12) and (13) for the Bodart data. To do this, we employ the Kolmogorov-Smirnov test for the equality of two distribution functions as described in Ref. 15.

The procedure for applying the Kolmogorov-Smirnov test is to find the maximum absolute difference D between the empirical distribution functions, and to multiply this by the factor $\sqrt{mn/(m + n)}$, where m and



(a) Battles won by attacker
(612 battles in sample)

(b) Battles won by defender
(468 battles in sample)

Fig. 9—Bodart-data-set distribution of attacker and defender casualty fractions

n are the sample sizes for the two empirical distributions. If we set

$$w = D\sqrt{mn/(m + n)},$$

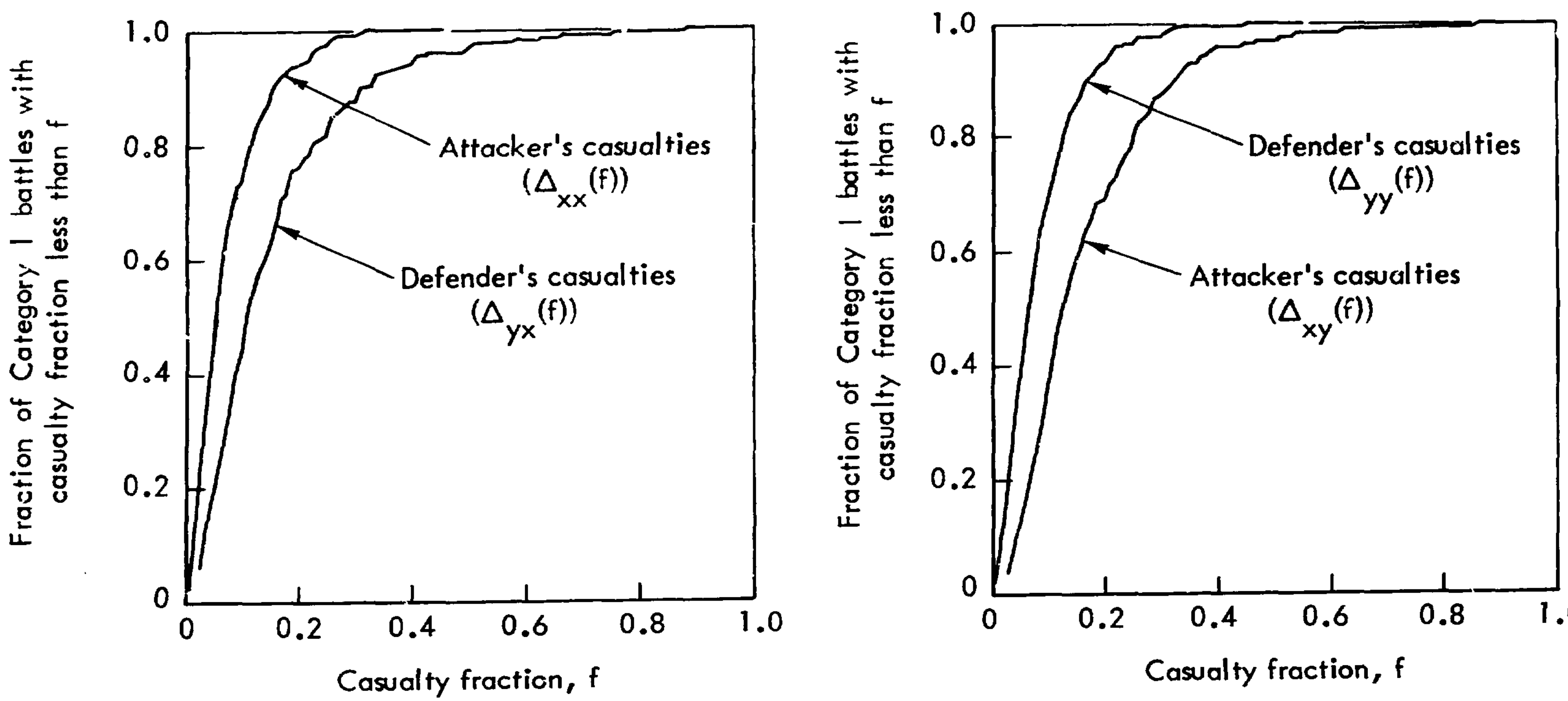
then

$$1 - K(w) = 1 - \sum_{j=-\infty}^{j=\infty} (-1)^j e^{-2j^2w^2}$$

is the probability, under appropriate asymptotic conditions, that the deviation between two empirical distribution functions will be more than w, given that the empirical distribution functions actually are obtained from independent random samples from a common continuous distribution function.

Applying this procedure to the data represented in Fig. 9 yields a w of about 1.22 and, referring to a table of K(w) values given in Ref. 15, we find that a deviation greater than 1.22 would occur by chance about 10 percent of the time, given that the empirical distributions corresponding to $\Delta_{xy}(u)$ and $\Delta_{yx}(u)$ are actually from a common distribution. Comparable results are obtained for the empirical distributions corresponding to $\Delta_{xx}(u)$ and $\Delta_{yy}(u)$. These results are taken to indicate that we may reasonably proceed on the assumption that Eqs. (12) and (13) hold for the Bodart data. Even if a strict equality does not hold between the distribution functions involved in Eqs. (12) and (13), a comparison of (a) and (b) of Fig. 9 shows that it would be unreasonable to believe that the difference could be very great.

Next, we proceed to analyze the Category I (open) battles, as a group separate from the Category II (closed) battles. Figure 10 gives the empirical distribution of casualty fractions for these battles. Applying the Kolmogorov-Smirnov test to the empirical distributions corresponding to Δ_{xx} and Δ_{yy} yields a w of about 1.35. A larger deviation than this would occur by chance about 5 percent of the time if the two empirical distributions were actually from a common distribution. For the empirical distributions corresponding to Δ_{xy} and Δ_{yx} a



(a) Battles won by attacker
(518 battles in sample)

(b) Battles won by defender
(415 battles in sample)

Fig. 10 — Distribution of attacker and defender casualty fractions for Category I (open) battles

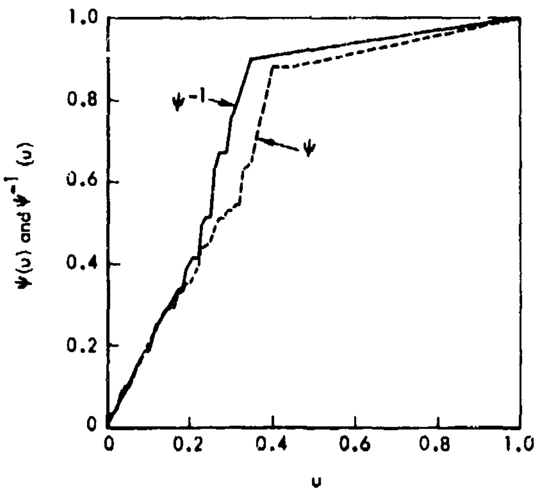


Fig. 11—ψ and ψ^{-1} functions for Category I battle data

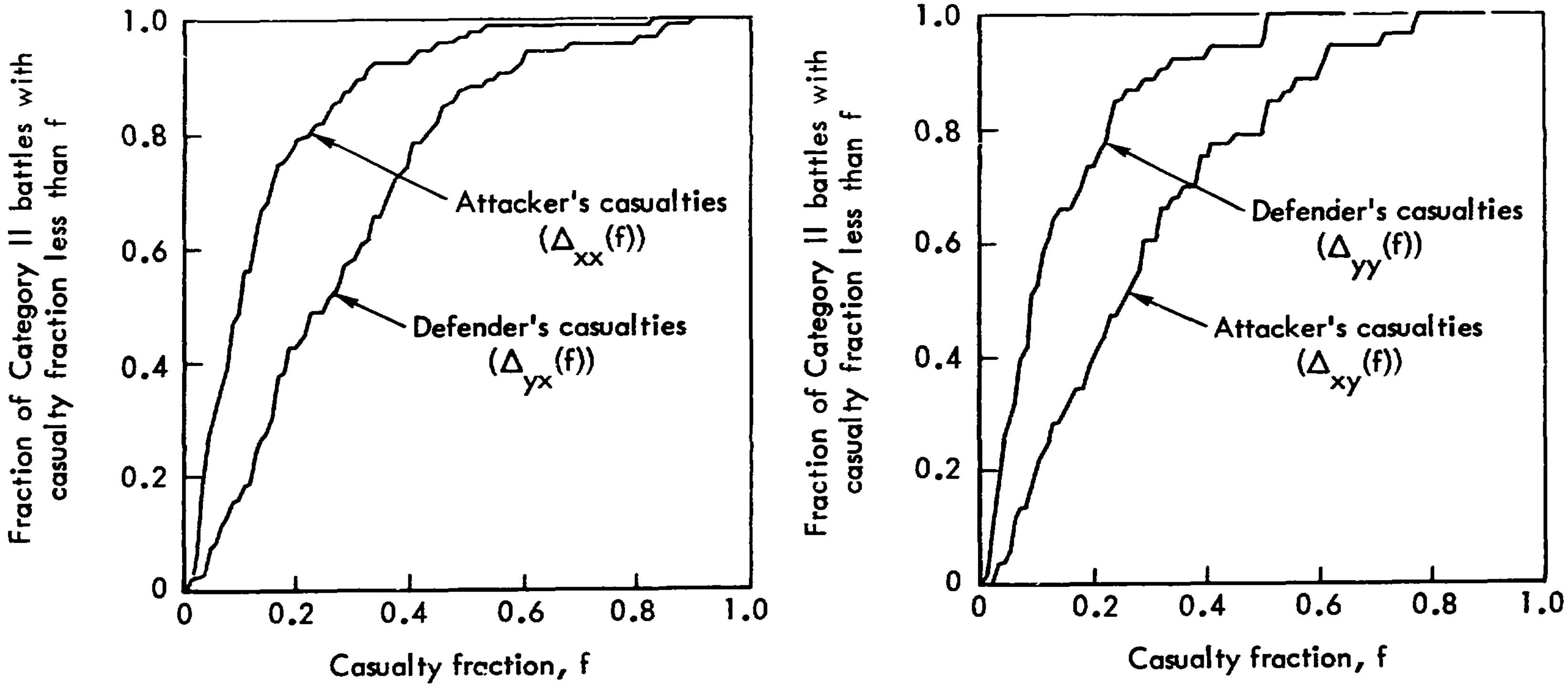
value of w = 1.44 is obtained. A larger value than this would occur by chance about 3 percent of the time if the two empirical distributions were actually from a common distribution. We do not consider these probabilities so small as to cause us to discard Eqs. (12) and (13) for the Category I battle data in the present context. Independently of this opinion, the Ψ and Ψ^{-1} functions for the Category I data are certainly not much different, as is evidenced by the curves of Fig. 11, and the breakpoint hypothesis would come to grief in any case.

The empirical distribution curves for the Category II data are shown in Fig. 12. For the empirical distributions corresponding to Δ_{xx} and Δ_{yy}, w is about 0.38 and would be exceeded by chance about 99.9 percent of the time. For the empirical distributions corresponding to Δ_{xy} and Δ_{yx}, w is about 0.47 and would be exceeded by chance about 98 percent of the time. Clearly there is no reason in these results for rejecting Eqs. (12) and (13). Consequently, the breakpoint hypothesis does not hold for the Category II data set either.

Although we shall not present the details, an application of the Kolmogorov-Smirnov test to the empirical casualty-fraction distributions of Fig. 7 shows that their deviation from Eqs. (12) and (13) is acceptable (a larger deviation would occur by chance about 30 percent of the time). Thus, while Eqs. (12) and (13) hold to a reasonable degree of approximation for all of the data analyzed, nevertheless

$$\Psi = \Psi^{-1} \neq I,$$

and this is in direct contradiction to Lemma 1. As a result, the breakpoint hypothesis is untenable. The same conclusion (that the breakpoint



(a) Battles won by attacker (94 battles in sample)

(b) Battles won by defender (53 battles in sample)

Fig. 12—Distribution of attacker and defender casualty fractions for Category II (closed) battles

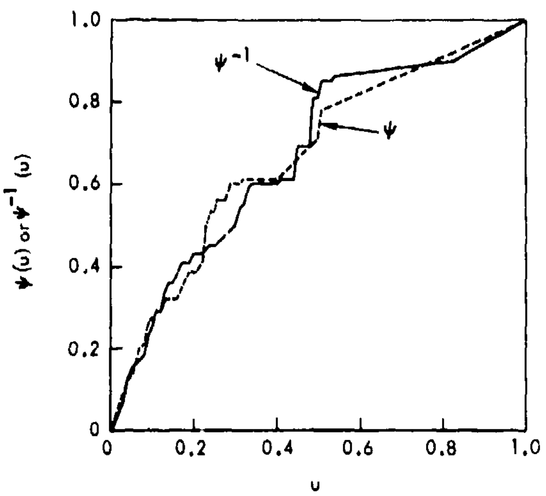


Fig. 13—Ψ and Ψ^{-1} functions for Category II battle data

hypothesis is untenable) can, of course, be demonstrated without the use of Lemma 1 and without using Eqs. (12) and (13). The procedure is simply to construct Figs. 11 and 13 by the graphical procedure explained earlier, and to observe that the resulting Ψ and Ψ^{-1} functions clearly are not mathematically inverse functions. However, the method used above, which invokes Eqs. (12) and (13) may shed light on the manner in which the breakpoint hypothesis fails.

We may sum up the results of this section by saying that for all of the data sets analyzed, Ψ and Ψ^{-1} are evidently *not* mutually inverse mathematical functions as is required by the breakpoint hypothesis. Consequently, the breakpoint hypothesis is untenable. In fact, rather than being inverse functions, it appears that

$$\Psi = \Psi^{-1} \neq I.$$

In addition, Eqs. (12) and (13) hold, at least approximately, for the data analyzed. This suggests that the reason why Ψ and Ψ^{-1} fail to obey the inverse functional relationship may be a consequence of the equality of the distribution functions expressed in Eqs. (12) and (13).

IV. SOME SPECULATIONS AND SUGGESTED APPROACHES

We have shown in the foregoing that the breakpoint hypothesis in the form given earlier predicts that a certain pair of functions are mathematical inverses of each other, but that the empirical determinations of these functions plainly do *not* exhibit any inverse functional relationship. Consequently, the breakpoint hypothesis fails. We now offer some speculations and tentative suggestions for future work on the causes of this observed contradiction between theory and fact. It will be helpful to review the statement of the breakpoint hypothesis in its original form before proceeding. It states that

1. Each side selects independently a breakpoint from a distribution of such breakpoints and gives up the battle when its casualty fraction reaches its breakpoint (Hypothesis A).
2. These breakpoint distribution curves are generally applicable (Hypothesis B).
3. The casualty fractions of the forces are deterministically and monotonically related to each other via the Ψ function (Hypothesis C); i.e.,

$$f_x(t) = \Psi[f_y(t)], \qquad 0 \le t \le T.$$

In the following sections we shall consider some tentative modifications of this breakpoint hypothesis and discuss them in terms of the light they shed on the prospects for developing a theory that will satisfactorily explain the extant data. Since devising a theory that would account for most of the more important facts regarding the battle termination process is beyond the scope of the present study, the observations and speculations put forward are incomplete. They are offered in the hope that subsequent investigations of battle termination phenomena may find some of these suggestions helpful.

FIRST MODIFICATION OF THE BREAKPOINT HYPOTHESIS

Lemma 2 shows that under the hypotheses set forth above there are only three possible relations between the casualty-fraction

distributions. If $\Psi(s) \geq s$ for all s, then

$$\Delta_{yy}(s) \geq \Delta_{xy}(s),$$

and

$$\Delta_{xx}(s) \leq \Delta_{yx}(s),$$

for all s. If $\Psi(s) \leq s$ for all s, then

$$\Delta_{yy}(s) \leq \Delta_{xy}(s),$$

and

$$\Delta_{xx}(s) \geq \Delta_{yx}(s),$$

for all s. If $\Psi(s)$ is alternately larger and smaller than s, then the theoretical casualty-fraction distributions alternately loop above and below each other. It is plain that the empirical casualty-fraction distributions do not exhibit either of these three behaviors, and this constitutes yet another conflict of the breakpoint hypothesis and available data. However, it also suggests a way to evade the difficulty. It involves using one Ψ function when the attacker wins, and a different Ψ function when the defender wins. Thus, Hypothesis C is modified to the extent of allowing the Ψ function to depend on the particular battle in a conceptually simple way. Based on the empirical findings expressed in Figs. 8, 11, and 13, it appears that we should consider Hypothesis D.

Hypothesis D. There is a monotone nondecreasing function Ψ such that

$$f_x = \Psi(f_y)$$

when the defender wins, while

$$f_x = \Psi^{-1}(f_y)$$

when the attacker wins; moreover, $\Psi(s) \geq s$ for all s. The Ψ function referred to in Hypothesis D should not be confused with the Ψ function of Hypothesis C. While they may be numerically similar, they are conceptually quite distinct.

Now, when we retrace the proof of Lemma 2, making appropriate changes to reflect Hypothesis D instead of Hypothesis C, we find that the conclusions change to read as follows:

When the defender wins,

$$\Delta_{yy}(s) \geq \Delta_{xy}(s) \tag{14}$$

for all s. When the attacker wins,

$$\Delta_{xx}(s) \geq \Delta_{yx}(s) \tag{15}$$

for all s. These relations are certainly encouraging, since they are qualitatively consistent with the data of Figs. 7, 9, 10, and 12. If we interpret the results of the Kolmogorov-Smirnov tests made earlier for the deviation between empirical casualty distributions to mean that Eqs. (12) and (13) hold, it is statistically proper to increase our sample size by merging the data for empirical distribution curves Δ_{yy} and Δ_{xx}, and also for empirical distribution curves Δ_{xy} and Δ_{yx}. When we do that, it is convenient to let ℓ stand for "loser" and w stand for "winner" and to write the relations (14) and (15) in the single form

$$\Delta_{ww}(s) \geq \Delta_{\ell w}(s) \tag{16}$$

for all s; and Hypothesis D under the same conditions can be rewritten as

$$f_{\ell} = \Psi(f_w),$$

with $\Psi(s) \geq s$ for all s.

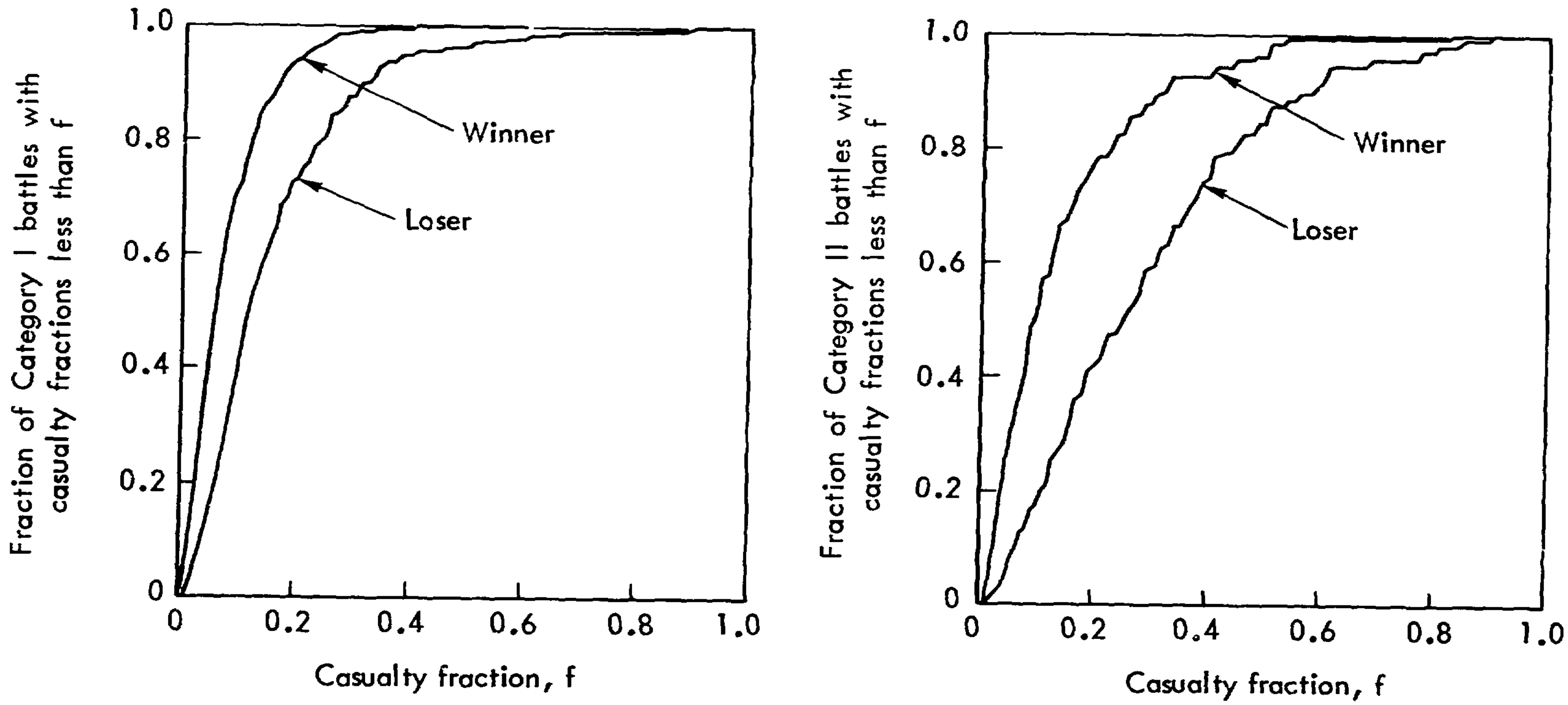
A little thought shows that we may apply the graphical procedure depicted in Fig. 5 to the empirical distribution fucntions Δ_{ww} and

$\Delta_{\ell w}$ to obtain estimates of the Ψ function. To see how well this works out, we first plot the loser's and the winner's casualty-fraction distributions for Category I and Category II battles as in Fig. 14. The values obtained for the function Ψ as determined from the distributions of loser's and winner's casualty fractions are shown in Fig. 15.

As can be seen from Fig. 15, the empirically determined Ψ functions for the Category I and the Category II data are about equal, and both are roughly linear up to argument values of approximately 0.5. In fact, we have roughly that

$$\Psi(u) = \begin{cases} 2u & \text{for } 0 \le u \le 0.5 \\ 1 & \text{for } 0.5 \le u \le 1 \end{cases} .$$

Suppose we ask what break curves, when taken with this value of Ψ, would reproduce the observed winner's and loser's casualty fractions shown in Fig. 14. Because of the quasi-exponential shape of the observed casualty-fraction distribution curves, suppose we limit ourselves to exponential break curves, as discussed in Example 3 of Appendix E. After some trial-and-error experimentation, we found that the break curves shown in Fig. 16 would produce the tentative theoretical fits shown in Fig. 17. The qualitative agreement between the theoretical and observed casualty distribution curves is very encouraging. However, the quantitative agreement, particularly for the Category I battles in the range of casualty-fraction values between 0 and 0.1 or 0.2, is not very good. Application of the Kolmogorov-Smirnov test procedure indicates that a worse agreement due to chance alone could be expected for the Category II data about 10 percent of the time for the loser's casualty fraction, and about 30 percent of the time for the winner's casualty fraction. For the Category I data, a poorer agreement due to chance alone would be expected only about 2 percent of the time for the winner's casualty fraction and hardly ever for the loser's casualty fraction. Under these circumstances, it is reasonable to take the position that a $\Psi(u) = 2u$ function and exponential break curves may not provide a satisfactory explanation of the observed data. Presumably the trial-and-error fitting process could be improved



(a) Category I battles
(933 battles in sample)

(b) Category II battles
(147 battles in sample)

Fig. 14—Distribution of winner and loser casualty fractions

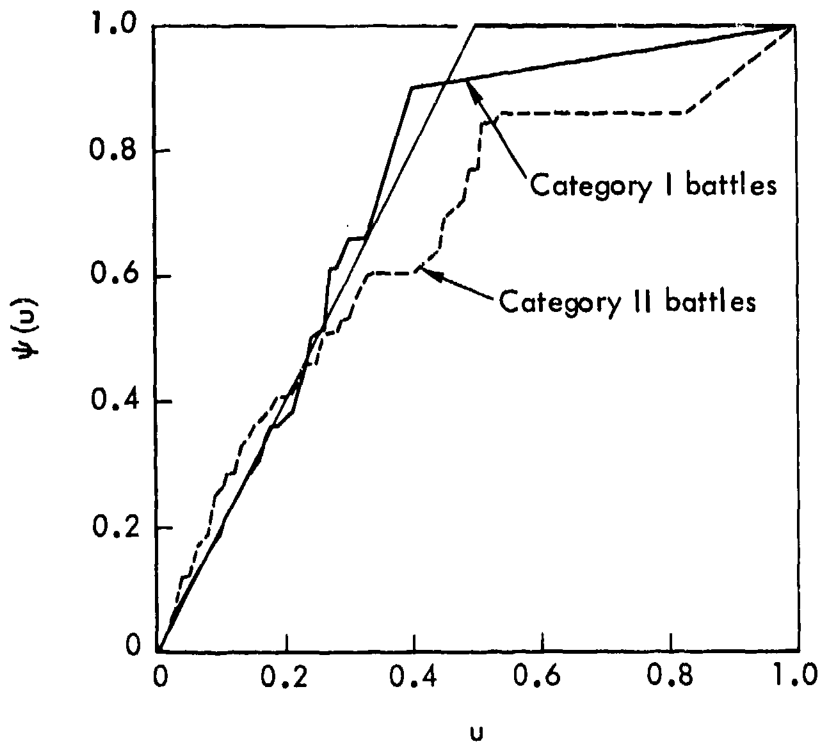
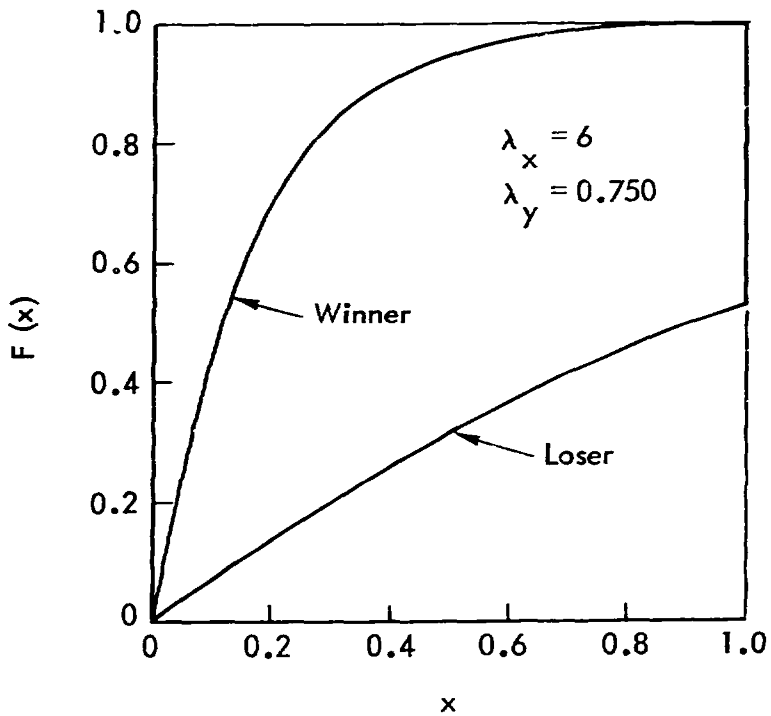
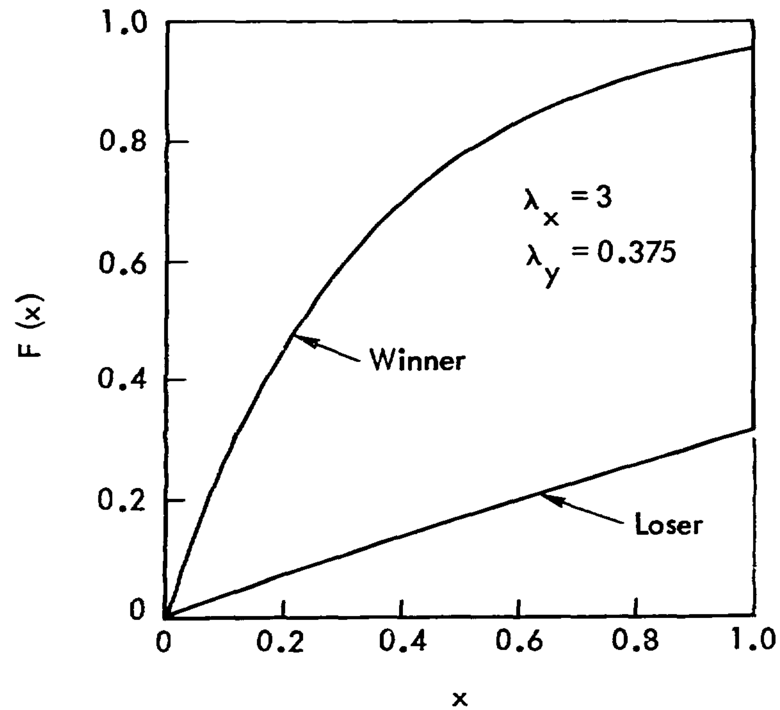


Fig. 15—ψ functions from loser's and winner's casualty - fraction distributions

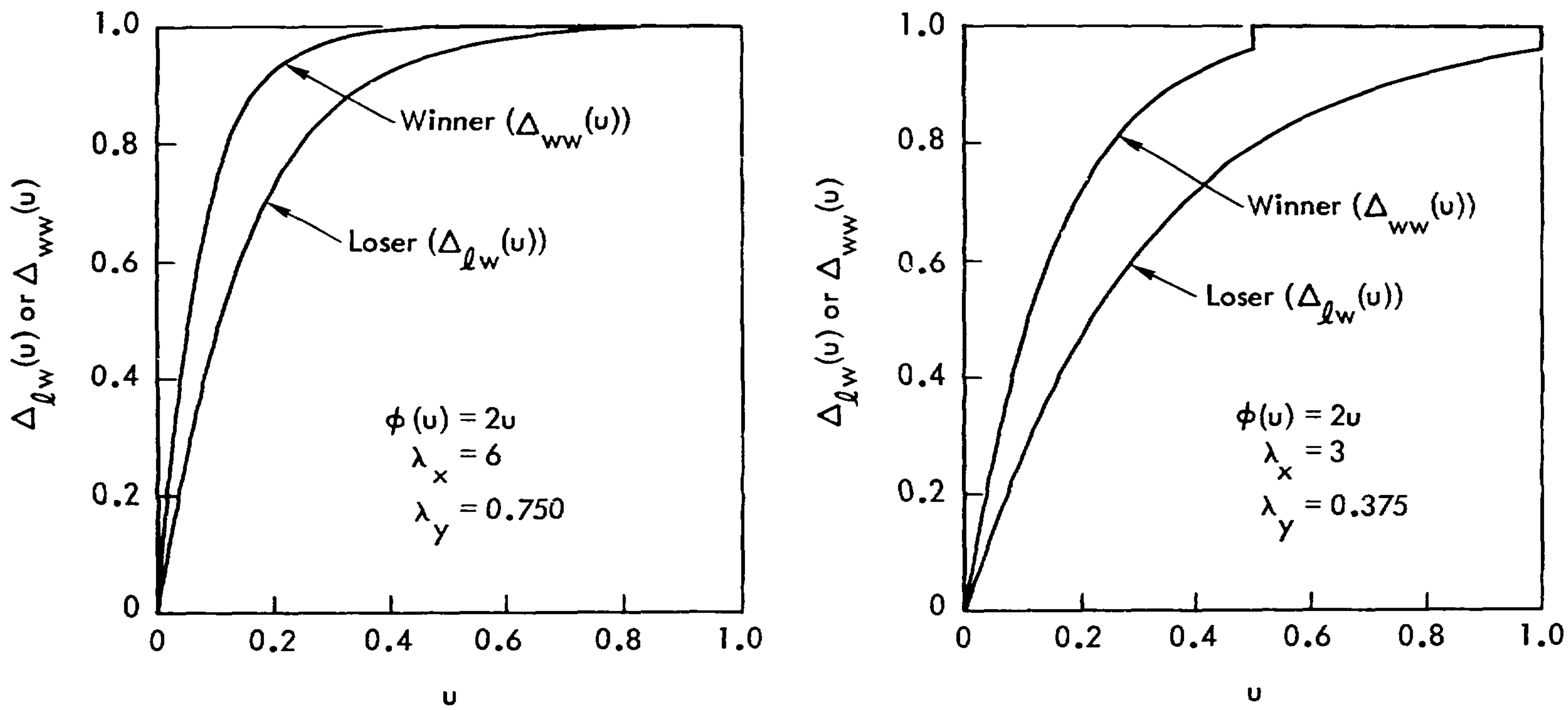


(a) Category I battles



(b) Category II battles

Fig.16—Assumed break curves



(a) Category I battles

(b) Category II battles

Fig. 17—Tentative theoretical fit to winner's and loser's casualty-fraction distributions

somewhat by more formal methods for fitting theoretical to empirical curves. The fit might also be improved by rescaling the exponential break curves as described in Example 3 of Appendix E (see footnote, p. 79). To get a curve with the same general behavior as that of the loser's break curve in part (a) of Fig. 16, it would be permissible to take λ_ℓ negative, since the rescaling procedure will ensure that the resulting function is a proper cumulative probability distribution. These very interesting possibilities could not be pursued in adequate depth within the scope of this investigation. Whether the fit would be improved to a satisfactory level is impossible to say with complete assurance without actually trying it. It may also be possible to fit the observed data by permitting the break curves to deviate from strict exponentiality, especially for the lower casualty-fraction values. However, the number of free parameters then available for fitting the data may be excessive, reducing the degrees of freedom and interfering with the power of the statistical procedures to detect genuine departures from the null hypothesis.

Even if a good fit to the data can be obtained by assuming that Ψ has the form

$$\Psi(u) = \begin{cases} 2u, & 0 \le u \le \frac{1}{2} \\ 1, & \frac{1}{2} \le u \le 1 \end{cases}$$

and carefully adjusting the break curves, it would still make sense to determine whether or not the casualty-fraction data satisfy the relation

$$f_\ell = \Psi(f_w).$$

If they do not satisfy this relation, then the situation may be much more complicated than the apparent good fit to the loser's and winner's casualty-fraction distribution might indicate. In order to determine whether or not there is an approximately linear relationship between the loser's and the winner's casualty fractions as implied by Hypothesis D

and Fig. 15, we first consider some data on casualty fractions for various battles from Ref. 8. These data are plotted in Fig. 18. The statistics for a linear regression of $\ln f_{\ell}$ on $\ln f_{w}$ were computed using these data. They lead to an estimated regression line given by

$$\ln f_{\ell} = -0.342 + 0.708 \ln f_{w}.$$

A more complete list of statistics for this regression computation is included in Table 3.

It is evident from Fig. 18 that the functional relationship supposed by Hypothesis D cannot be true--or, more precisely stated, the implied assumption that we have made in the way we have used the Ψ function (namely, that it is the same for all battles) is not valid. Of course, our use of Hypothesis C also made essential use of the tacit assumption that the function Ψ as defined in that hypothesis was the same for all battles. Indeed, in changing from Hypothesis C to Hypothesis D, one of the motivations was to permit the Ψ function to vary depending on whether the attacker or the defender won the battle, and thus to allow at least that degree of variation in the Ψ function from one battle to another. However, it is clear from Fig. 18 that there is more than just that amount of interbattle variability. When the linear-regression computations are performed for the Category I and Category II data, and for log-transformed casualty fractions or untransformed casualty fractions, the results are as given in Tables 4 and 5. The significant observation is that in none of these cases does the regression curve approximate very closely the expected $\Psi(u) = 2u$ function, except possibly for small values of the casualty fraction. Thus, it appears that not only do the casualty fractions vary more widely from battle to battle than envisioned in Hypothesis D, but they do not even on the average follow the relation anticipated on the basis of the empirical determination of the Ψ function presented in Fig. 15.

Consequently, the validity of Hypothesis D is uncertain. One other approach might be tried here--that is, to assume the Ψ function to be actually as indicated in the results of the regressions of f_{ℓ} on f_{w} (or of the corresponding log-transformed quantities), and then

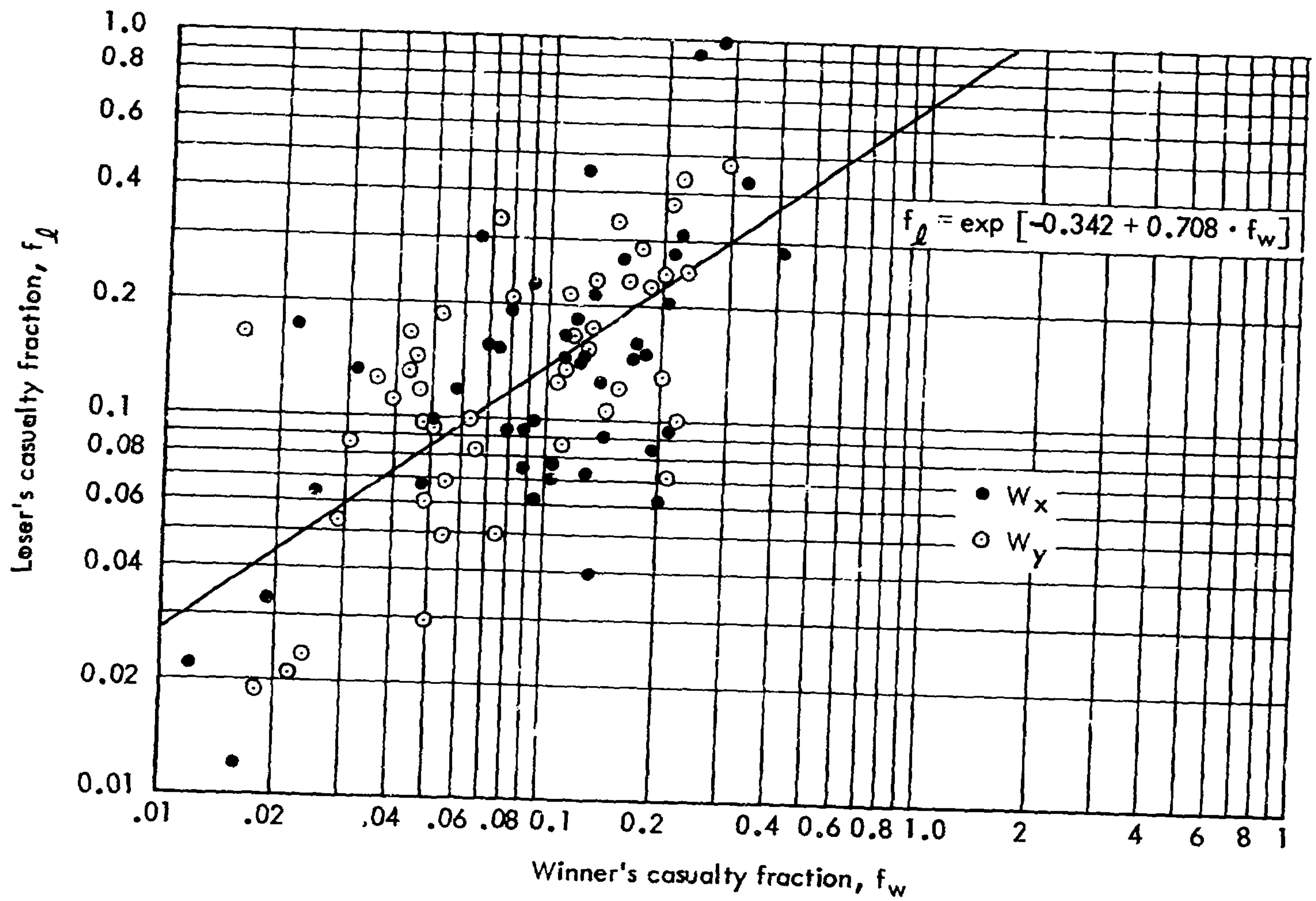


Fig. 18—Casualty fractions, f_ℓ versus f_w

Table 3

STATISTICS FOR REGRESSION OF $\ln f_\ell$ ON $\ln f_w$[a]

Item	Value
Regression-line intercept, a	-0.34217
Regression-line slope, b	0.70753
Standard deviation of slope, s sub b	0.0805
Correlation coefficient, r	0.6796
Standard deviation of estimate, $s_{y\|x}$	0.6364
Variation of estimate, $s_{y\|x}$**2	0.4050
Mean of X-values, m(X)	-2.4425
Standard deviation of X-values, S.D.(X)	0.8286
Mean of Y-values, m(Y)	-2.0562
Standard deviation of Y-values, S.D.(Y)	0.8627

[a]Regression model: $\ln(f_\ell) = a + b \ln(f_w)$. Number of data points = 92.

Table 4

RESULTS OF COMPUTATION FOR LINEAR REGRESSION OF $\ln f_\ell$ ON $\ln f_w$[a]

Item	Battle Category	
	I (open)	II (closed)
Sample size	933	147
Intercept, a	-1.1480	-0.5690
Slope, b	0.3484	0.4077
Standard deviation of slope, σ_b	0.0271	0.0650
Correlation coefficient, r	0.3883	0.4621
Standard deviation of estimate, $s_{y\|x}$	0.7751	0.7762
Mean of $x = \ln f_w$	-2.9358	-2.4076
Standard deviation of x	0.9370	0.9885
Mean of $y = \ln f_\ell$	-2.1708	-1.5506
Standard deviation of y	0.8407	0.8723
Var x	0.8779	0.9771
Var y	0.7068	0.7608
r^2	0.1508	0.2135
$1 - r^2$	0.8492	0.7865

[a]Regression model: $\ln f_\ell = a + b \ln f_w$.

Table 5

RESULTS OF COMPUTATION FOR LINEAR REGRESSION OF f_ℓ ON f_w[a]

Item	Battle Category	
	I (open)	II (closed)
Sample size	933	147
Intercept, a	0.1163	0.1904
Slope, b	0.5358	0.6851
Standard deviation of slope, s_b	0.0634	0.1080
Correlation coefficient, r	0.2668	0.4659
Standard deviation of estimate, $s_{y\|x}$	0.1305	0.1769
Mean of $x = f_w$	0.0773	0.1397
Standard deviation of x	0.0674	0.1355
Mean of $y = f_\ell$	0.1577	0.2861
Standard deviation of y	0.1353	0.1992
Var x	0.0045	0.0184
Var y	0.0183	0.0397
r^2	0.0712	0.2171
$1 - r^2$	0.9288	0.7829

[a]Regression model: $f_\ell = a + b f_w$.

see if some simple and reasonable forms of the break curves approximately reproduce the observed data. The scope of this investigation did not permit this suggestion to be pursued.

It is worth noting a couple of features of the breakpoint-hypothesis modification characterized by Hypothesis D. First, Hypothesis D requires that the loser and the winner be identified by some means extraneous to the model, and this precludes use of the modified breakpoint model for predicting the winner. This is a serious drawback in terms of the conventional uses of breakpoint-type hypotheses. Once the winner has been determined, however, Hypothesis D could still be used to find the casualty fractions on both sides. Once the winner has been identified, perhaps by methods similar to those discussed in Ref. 16, this can be done simply by entering the break curves (such as those of Fig. 16, for example) with a random number on the ordinate, and reading off the corresponding casualty-fraction values for the winner and loser from the abscissa.

The second feature is that if we adopt Hypothesis D and also assume that the Ψ functions for the Category I and the Category II battles are identical, as suggested by Fig. 15, then it can be shown that Hypothesis B is untenable. The argument goes as follows. Suppose Hypothesis D holds, that the Ψ function for Category II is the same as that for Category I, and that the break curves for Category II are the same as those for Category I. Then the empirical casualty-fraction distributions for Category II would have to be the same as those for Category I. But a superficial inspection of (a) and (b) of Fig. 14 reveals that the empirical casualty-fraction distributions for Category II are not the same as those for Category I. Application of the Kolmogorov-Smirnov test to these data confirms the common-sense observation that deviations from equality at least as large as the ones observed would occur by chance alone only about once in 10^7 times. This suggests that Hypothesis B may be untenable at least to the extent of requiring that a distinction be maintained between Category I and Category II battles.

In Table 5 the average of f_ℓ is almost exactly twice the average of f_w. This is of some interest, since by (b) of Fig. 14 $\Psi(u)/u$ is also very nearly equal to 2 for $0 \le u \le \frac{1}{2}$. However, since the connection between

these two facts is not clear, the numerical agreement between these two quantities may be purely coincidental. A further exploration of the potential relation between these values was not within the scope of this study.

Another fact, of more interest, since it is apparently more closely related to this modification of the breakpoint hypothesis, is a finding of Schmieman.[13] To describe this finding it is necessary to understand that past investigators of combat data,[8,10] as well as Schmieman, have devised "advantage" parameters that, on the basis of certain theoretical considerations, ought to be related to the winning or losing of an engagement. All the advantage parameters that have been introduced to date are functions of the casualty-fraction values only but depend on which side's casualty fraction is chosen as the first variable. One of the criteria of whether an advantage parameter adequately reflects the relative advantage of the opposing forces in a battle is whether or not its value agrees with the side that is observed to win the battle. References 8 and 10, basing the selection of casualty-fraction values on which side was the attacker and which the defender, found agreements of 74 percent and 78 percent between the resulting advantage-parameter values and the observed winning side. Schmieman has confirmed this, finding an agreement of 79 percent between advantage-parameter values and observed winners for this case. However, Schmieman went on to consider two other cases, viz., when the casualty fractions are taken as those of the larger and smaller force, respectively; and when the casualty fractions are taken as those of the winning and the losing side, respectively. When casualty-fraction values were taken as those of the larger and the smaller force, respectively, Schmieman again found an agreement of 79 percent between the advantage-parameter values and the observed winner. However, when he took casualty-fraction values as those of the winner and the loser, respectively, the agreement between the advantage-parameter value and the observed winner jumped to 97 percent, or nearly perfect agreement. The modification of the breakpoint hypothesis discussed in this section qualitatively accounts for this phenomenon. An exploration of the quantitative agreement between Hypothesis D and Schmieman's discovery was not within the scope of this study.

SECOND MODIFICATION OF THE BREAKPOINT HYPOTHESIS

Following the discussion of Lanchester's square-law equations as given in Ref. 8, we write

$$dx/dt = -Dy$$

$$dy/dt = -Ax,$$

where x and y are the attacker's and defender's remaining troop strength, respectively. By division, it follows that

$$\mu^2 = \frac{1 - a^2}{1 - d^2},$$

where

$$\mu^2 = (D/A)(y_0^2/x_0^2),$$

$$a = 1 - f_x,$$

and

$$d = 1 - f_y.$$

Consequently, we may solve for f_x in terms of f_y and μ as

$$f_x = 1 \pm \sqrt{1 - \mu^2(2f_y - f_y^2)}.$$

Here μ is a positive constant* as long as A and D are positive. It is an index of the defender's advantage in the sense that if $\mu > 1$, f_x reaches unity before f_y does, while if $\mu < 1$, f_y reaches unity before

*That is, μ is independent of the battle-time variable, t. But it may take on different values in different battles.

f_x does. When A and D are both positive, obviously f_x must be an increasing function of f_y, and this fact dictates the choice of the negative sign in the Lanchester relation connecting f_x and f_y, i.e., we have

$$f_x = 1 - \sqrt{1 - \mu^2(2f_y - f_y^2)}.$$

Accordingly, the corresponding functional relationship, φ, is defined by the equation

$$\varphi(u) = 1 - \sqrt{1 - \mu^2(2u - u^2)},$$

and

$$\varphi^{-1}(v) = 1 - \sqrt{1 - \mu^{-2}(2v - v^2)}.$$

Expanding the expression for $\varphi(u)$ in a Maclaurin series yields

$$\varphi(u) = \tfrac{1}{2}\mu^2(2u - u^2) + \tfrac{1}{2}\mu^4 u^2 + \text{HOT},$$

where HOT stands for *higher-order terms*. Simplifying, we can write

$$\varphi(u) = \mu^2 u + \text{HOT}.$$

If we linearize by neglecting the HOT, then we have approximately

$$\varphi(u) = \gamma u$$

with $\gamma = \mu^2$.

The point to this simplification is that we know something about the distribution of μ^2 from previous work(8,10) and hence about the distribution of γ. Specifically, $\ln\mu$ is a stochastic function of the force ratio x_0/y_0, approximately defined by(8)

$$\ln\mu = 0.115 - 0.367 \ln(x_0/y_0) + \eta,$$

where η is a normally distributed random variable independent of (x_0/y_0), having mean and standard deviation approximately equal to 0.297. The force ratio itself is log-normally distributed, the mean of $\ln(x_0/y_0)$ being about 0.156, with a standard deviation of about 0.156. Using these data with the rules for obtaining the distribution of sums of independent normally distributed random variables, we find that $\ln\mu$ is normally distributed, with mean value about 0.057 and standard deviation about 0.350. These values are based on a sample size of 92, so the standard deviation of the mean is about equal to

$$0.350/\sqrt{92} \cong 0.04,$$

and we see that the mean is not significantly different from zero. In what follows, we will, for simplicity, take $\ln\mu$ as being normally distributed, with zero mean and a standard deviation of about 0.35. Then, of course, $\ln\gamma = 2\ln\mu$ will be normally distributed with zero mean and a standard deviation of about 0.7.

Thus, we expect, on the basis of the Lanchesterian model just introduced, to have

$$\ln f_x = \ln\gamma + \ln f_y,$$

with $\ln\gamma$ distributed as just discussed. A calculation using data from Ref. 8 shows that $\ln(f_x/f_y)$ has a mean value of about 0.116 and a standard deviation of 0.762. Again, the mean value is not significantly different from zero. These values are close to twice those cited above for $\ln\mu$, which is what we expect, since

$$\ln(f_x/f_y) = \ln\gamma = 2\ln\mu.$$

Accordingly, our linearization of φ by neglecting higher-order terms in its series expansion appears to be an acceptable approximation. We assume in what follows that $\ln\gamma$ is normally distributed with mean zero and a standard deviation of about 0.76.

This amounts to introducing another version of Hypothesis C, which we write out explicitly as Hypothesis E.

<u>Hypothesis E</u>. The casualty fractions of the forces engaged in a given battle are related to each other according to the following relation:

$$f_x = \gamma f_y,$$

where γ is a constant for any particular battle but is a random variable that varies from battle to battle in a log-normal distribution with parameters (0, 0.76).

With this interpretation of γ as a log-normally distributed random variable, Eqs. (E-1) through (E-4) of Appendix E must be considered as giving formulae for conditional probabilities, given a particular value of γ. To emphasize this conditional dependence on γ, we shall write the right-hand side of Eq. (F-2) as $\Delta_{yx}(q;\, \lambda_x,\, \lambda_y,\, \gamma)$, so that

$$P(f_y < q|W_x) = \Delta_{yx}(q;\, \lambda_x,\, \lambda_y,\, \gamma),$$

where the left-hand side must be thought of as the defender's conditional casualty-fraction distribution, given an attacker win and a particular value of γ. The usual transposition gives the dual formula. From Eq. (E-4), we find

$$\begin{aligned} P(f_x < s|W_x) &= \Delta_{yx}(\gamma^{-1}s;\, \lambda_x,\, \lambda_y,\, \gamma)\sigma(\gamma,\, s) + \sigma(s,\, \gamma) \\ &= \text{(say) } \Delta_{xx}(s;\, \lambda_x,\, \lambda_y,\, \gamma), \end{aligned}$$

which is to be interpreted as a conditional casualty-fraction distribution, given γ and W_x. The usual transposition gives the dual relation.

To find the corresponding unconditional casualty-fraction distributions requires an integration with respect to the probability element of γ. Thus, for example, we shall write

$$\begin{aligned} P(f_y < q|W_x) &= \frac{1}{\sigma\sqrt{2\pi}} \int_0^\infty \Delta_{yx}(q;\, \lambda_x,\, \lambda_y,\, \gamma) \exp\, [-(\ln\gamma)^2/2\sigma^2]\gamma^{-1}\, d\gamma \\ &= \text{(say) } \Delta_{yx}(q;\, \lambda_x,\, \lambda_y), \end{aligned}$$

where $\sigma = 0.76$. Performing the usual transposition and then making the change $u = \gamma^{-1}$ in the variable of integration yields

$$P(f_x < q|W_y) = \Delta_{xy}(q;\ \lambda_y,\ \lambda_x).$$

Analogously, we have

$$P(f_x < s|W_x) = \Delta_{xx}(s;\ \lambda_x,\ \lambda_y)$$

and

$$P(f_y < s|W_y) = \Delta_{yy}(s;\ \lambda_y,\ \lambda_x),$$

where

$$\Delta_{xx}(s;\ \lambda_x,\ \lambda_y) = \frac{1}{\sigma\sqrt{2\pi}}\int_0^\infty \Delta_{xx}(s;\ \lambda_x,\ \lambda_y,\ \gamma)\ \exp\ [-(\ln\gamma)^2/2\sigma^2]\gamma^{-1}\ d\gamma,$$

with $\sigma = 0.76$.

Tables 6 through 11 present casualty-fraction distributions, computed on the basis of exponential break curves and Hypothesis E, for selected values of λ_x and λ_y. The numerical integration procedure employed is such that the second significant digit may occasionally be in error. Tables 7 and 9 are duals of each other, as expected. Casualty-fraction distributions for $\lambda_x = 1$, $\lambda_y = 3$ are obtainable by forming the dual of Table 10.

One noticeable feature of these theoretical results is that the casualty-fraction distribution of the winner would plot on a graph (one like part (a) of Fig. 14, for example) either below and to the right of the loser's casualty-fraction distribution or, at any rate, only very slightly above and to the left of it. Although the full range of parameter values could not be explored within the scope of this investigation, the results available to date suggest that this property is characteristic of the theoretical casualty-fraction distribution curves

Table 6

SOME THEORETICAL CASUALTY-FRACTION DISTRIBUTIONS FOR HYPOTHESIS E

q	$\lambda_x = 1$, $\lambda_y = 1$: $\Delta_{zz}(q;\lambda_x,\lambda_y)$[a]	$\lambda_x = 2$, $\lambda_y = 2$: $\Delta_{zz}(q;\lambda_x,\lambda_y)$[a]
.000	.000	.000
.025	.059	.111
.050	.115	.208
.075	.167	.293
.100	.216	.368
.125	.261	.435
.150	.304	.493
.175	.344	.546
.200	.382	.592
.225	.418	.634
.250	.451	.671
.275	.481	.704
.300	.511	.734
.325	.538	.761
.350	.564	.785
.375	.588	.807
.400	.610	.826
.425	.631	.844
.450	.651	.860
.475	.670	.874
.500	.688	.887
.600	.748	.928
.700	.796	.956
.800	.834	.975
.900	.865	.988
1.000	1.000	1.000
$P(W_z)$[b]	0.5	0.5

[a]For $\lambda_x = \lambda_y$, $\Delta_{xx}(q;\lambda_x,\lambda_y) = \Delta_{yy}(q;\lambda_x,\lambda_y) = \Delta_{xy}(q;\lambda_x,\lambda_y) = \Delta_{yx}(q;\lambda_x,\lambda_y)$.

[b]For $\lambda_x = \lambda_y$, $P(W_x) = P(W_y)$.

Table 7

THEORETICAL CASUALTY-FRACTION DISTRIBUTIONS

FOR HYPOTHESIS E, $\lambda_x = 1$, $\lambda_y = 2$[a]

q	$\Delta_{yx}(q;\lambda_x,\lambda_y)$	$\Delta_{xx}(q;\lambda_x,\lambda_y)$	$\Delta_{yy}(q;\lambda_x,\lambda_y)$	$\Delta_{xy}(q;\lambda_x,\lambda_y)$
.000	.000	.000	.000	.000
.025	.088	.090	.078	.090
.050	.169	.169	.149	.170
.075	.242	.240	.215	.241
.100	.309	.304	.275	.304
.125	.370	.362	.329	.361
.150	.426	.414	.380	.412
.175	.476	.461	.426	.458
.200	.522	.505	.468	.499
.225	.564	.544	.506	.536
.250	.603	.580	.542	.570
.275	.637	.612	.574	.601
.300	.669	.643	.604	.629
.325	.698	.670	.631	.654
.350	.724	.695	.656	.677
.375	.748	.719	.679	.698
.400	.770	.740	.700	.718
.425	.789	.760	.720	.735
.450	.808	.778	.737	.751
.475	.824	.795	.754	.766
.500	.839	.811	.769	.780
.600	.886	.863	.817	.823
.700	.919	.901	.852	.855
.800	.942	.930	.877	.878
.900	.958	.952	.894	.895
1.000	1.000	1.000	1.000	1.000

[a] $P(W_x) = 0.63$; $P(W_y) = 0.37$.

Table 8

THEORETICAL CASUALTY-FRACTION DISTRIBUTIONS

FOR HYPOTHESIS E, $\lambda_x = 1.5$, $\lambda_y = 3$

q	$\Delta_{yx}(q;\lambda_x,\lambda_y)$	$\Delta_{xx}(q;\lambda_x,\lambda_y)$	$\Delta_{yy}(q;\lambda_x,\lambda_y)$	$\Delta_{xy}(q;\lambda_x,\lambda_y)$
.000	.000	.000	.000	.000
.025	123	.128	.116	.129
.050	.230	.235	.218	.236
.075	.323	.327	.307	.327
.100	.405	.405	.385	.405
.125	.477	.473	.454	.472
.150	.539	.532	.514	.530
.175	.595	.583	.568	.580
.200	.643	.629	.615	.624
.225	.686	.668	.656	.663
.250	.723	.704	.693	.697
.275	.757	.735	.725	.727
.300	.786	.763	.754	.753
.325	.812	.788	.780	.777
.350	.834	.810	.802	.798
.375	.854	.830	.822	.817
.400	.872	.848	.840	.833
.425	.888	.864	.856	.848
.450	.902	.878	.870	.862
.475	.914	.891	.882	.874
.500	.925	.903	.893	.885
.600	.958	.940	.927	.918
.700	.978	.965	.948	.941
.800	.991	.983	.961	.956
.900	.999	.995	.969	.967
1.000	1.000	1.000	1.000	1.000

Table 9

THEORETICAL CASUALTY-FRACTION DISTRIBUTIONS
FOR HYPOTHESIS E, $\lambda_x = 2$, $\lambda_y = 1$[a]

q	$\Delta_{yx}(q;\lambda_x,\lambda_y)$	$\Delta_{xx}(q;\lambda_x,\lambda_y)$	$\Delta_{yy}(q;\lambda_x,\lambda_y)$	$\Delta_{xy}(q;\lambda_x,\lambda_y)$
.000	.000	.000	.000	.000
.025	.090	.078	.090	.088
.050	.170	.149	.169	.169
.075	.241	.215	.240	.242
.100	.304	.275	.304	.309
.125	.361	.329	.362	.370
.150	.412	.380	.414	.426
.175	.458	.426	.461	.476
.200	.499	.468	.505	.522
.225	.536	.506	.544	.564
.250	.570	.542	.580	.603
.275	.601	.574	.612	.637
.300	.629	.604	.643	.669
.325	.654	.631	.670	.698
.350	.677	.656	.695	.724
.375	.698	.679	.719	.748
.400	.718	.700	.740	.770
.425	.735	.720	.760	.789
.450	.751	.737	.778	.808
.475	.766	.754	.795	.824
.500	.780	.769	.811	.839
.600	.823	.817	.863	.886
.700	.855	.852	.901	.919
.800	.878	.877	.930	.942
.900	.895	.894	.952	.958
1.000	1.000	1.000	1.000	1.000

[a] $P(W_w) = 0.37$; $P(W_y) = 0.63$.

Table 10

THEORETICAL CASUALTY-FRACTION DISTRIBUTIONS

FOR HYPOTHESIS E, $\lambda_x = 3$, $\lambda_y = 1$[a]

q	$\Delta_{yx}(q;\lambda_x,\lambda_y)$	$\Delta_{xx}(q;\lambda_x,\lambda_y)$	$\Delta_{yy}(q;\lambda_x,\lambda_y)$	$\Delta_{xy}(q;\lambda_x,\lambda_y)$
.000	.000	.000	.000	.000
.025	.116	.099	.118	.110
.050	.214	.188	.217	.208
.075	.298	.267	.303	.295
.100	.370	.338	.377	.373
.125	.432	.402	.441	.442
.150	.486	.459	.498	.504
.175	.534	.510	.548	.559
.200	.575	.555	.592	.608
.225	.612	.596	.632	.652
.250	.645	.633	.667	.691
.275	.674	.665	.699	.726
.300	.701	.695	.727	.757
.325	.724	.721	.753	.784
.350	.745	.744	.776	.809
.375	.764	.765	.797	.831
.400	.781	.784	.817	.850
.425	.796	.801	.834	.868
.450	.810	.816	.850	.883
.475	.823	.830	.864	.897
.500	.834	.842	.877	.909
.600	.870	.879	.920	.946
.700	.895	.903	.950	.970
.800	.913	.919	.972	.985
.900	.926	.929	.989	.994
1.000	1.000	1.000	1.000	1.000

[a] $P(W_x) = 0.29$; $P(W_y) = 0.71$.

Table 11

THEORETICAL CASUALTY-FRACTION DISTRIBUTIONS

FOR HYPOTHESIS E, $\lambda_x = 6$, $\lambda_y = 0.75$[a]

q	$\Delta_{yx}(q;\lambda_x,\lambda_y)$	$\Delta_{xx}(q;\lambda_x,\lambda_y)$	$\Delta_{yy}(q;\lambda_x,\lambda_y)$	$\Delta_{xy}(q;\lambda_x,\lambda_y)$
.000	.000	.000	.000	.000
.025	.187	.156	.189	.166
.050	.326	.287	.329	.306
.075	.432	.397	.437	.422
.100	.516	.490	.524	.520
.125	.584	.567	.593	.602
.150	.638	.632	.651	.671
.175	.684	.687	.699	.728
.200	.721	.733	.739	.776
.225	.753	.771	.773	.816
.250	.780	.803	.802	.849
.275	.803	.830	.827	.877
.300	.823	.853	.849	.901
.325	.840	.872	.868	.920
.350	.855	.887	.885	.936
.375	.868	.901	.899	.950
.400	.879	.912	.912	.961
.425	.889	.921	.923	.970
.450	.898	.929	.934	.978
.475	.905	.935	.943	.985
.500	.912	.941	.951	.990
.600	.933	.955	.975	1.004
.700	.947	.961	.992	1.011
.800	.956	.965	1.003	1.014
.900	.963	.966	1.011	1.016
1.000	1.000	1.000	1.000	1.000

[a] $P(W_x) = 0.14$; $P(W_y) = 0.86$.

obtainable from Hypothesis E. However, the empirical data (see Fig. 14) indicate that the loser's casualty-fraction curve is distinctly to the right and below that of the winner, and so is in conflict with the consequences of Hypothesis E so far explored. This is not a very auspicious prospect for the second modification of the breakpoint hypothesis, and so it will not be further explored here.

THIRD MODIFICATION OF THE BREAKPOINT HYPOTHESIS

A third possible modification of the breakpoint hypothesis would be to give up Hypothesis B and permit the break curves themselves to vary depending on the class or type of battle that is under study. This modification will not be fully worked out here because it is beyond the scope of the study. However, some preliminary observations are offered.

The immediate question for this modification of the breakpoint hypothesis is whether or not the various break curves for the several types of battles can be combined to produce an aggregate or composite break curve valid for the whole universe of battles. For example, suppose that $F_i(x)$ is the break curve for battles of type i, and that battles of type i occur with relative frequency n_i/N, where N is the total number of battles in some data sample. Suppose we construct a new probability distribution function by a weighted-average break curve using the formula

$$F(x) = (1/N) \sum_{i=1}^{i=K} n_i F_i(x),$$

where K is the number of distinct types of battles. Is it true that the observed casualty-fraction distributions can be determined by F(x) without reference to the individual F_i? If so, then the original classification of battles into several types, each with its peculiar break curve would merely be introducing a distinction without a difference. However, a scan of the basic formulae in Appendix D shows no evident reason for believing that a simple average would produce a composite

break curve capable of generating casualty-fraction distributions applicable to the entire set of battles. Whether or not some other method of arriving at such a composite break curve would succeed is perhaps doubtful, but not completely resolved.

A second feature of a modified breakpoint hypothesis such as this is that (unless some composite break curve is appropriate--in which case the modification is no different from the original) confronting the hypothesis with empirical data requires subdividing the data into smaller groupings corresponding to the several battle types. This operation often reduces the ability of statistical tests to discriminate against the hypothesis when it is fallacious. For instance, consider the extreme case where each battle of the sample is supposed to belong to its own separate battle type--possibly on the romantic assumption that every battle is unique. Then there is no way in which the hypothesis can be disconfirmed by the data, and this is the case whether or not the hypothesis is valid. Less extreme cases than this tend to diminish, to a greater or lesser extent, the ability of statistical methods to detect an invalid hypothesis. As a result, anyone seriously proposing such a hypothesis should take care either to advance a hypothesis without too many distinct battle types, or to expand the size of the data sample in order to restore the sensitivity of the statistical procedure to a reasonable level.

There is an interestingly different and suggestive way of looking at breakpoint hypotheses of the sort represented by this modification. It can be introduced by considering that the process of classifying battles by type and hypothesizing distinct break curves for the various types will *ipso facto* generate a certain amount of stochastic dependence over the universe of all battles between the break level selected for one side and that selected for the opponent. Viewing the third modification of the breakpoint hypothesis as a special case of dependence between the casualty fractions on both sides (as distinct from the assumptions of Hypothesis A) may or may not be helpful in a practical sense, but it sheds light on the nature of the assumption being made.

MINIMAL REQUIREMENTS FOR A BATTLE TERMINATION THEORY

The properties that a satisfactory theory of battle termination should possess seem to include at least the following:

1. The theory should have a simplicity and "naturalness" of form in consonance with the principle of Ockham's Razor (William of Ockham, 1280-1349 A.D.) that "multiplicity ought not to be posited without necessity."

2. It must reproduce the observed quasi-exponential shape of empirical casualty distribution curves.

3. The winner's casualty-fraction distribution curve must lie above and to the left of the loser's casualty-fraction distribution curve, i.e.,

$$P(f_z|W_z) = \Delta_{zz} > \Delta_{z'z} = P(f_{z'}|W_z),$$

where z = x or y, and z' = y or x, respectively.

4. The theory must address the separate casualty distribution curves observed for the Category I and the Category II battles.

5. The theory must not produce an estimate of the Ψ function relating casualty fractions via the relations

$$f_x = \Psi(f_y)$$

or

$$f_\ell = \Psi(f_w)$$

that is at variance with the actual relations between these quantities.

6. The theory ought to explain why the loser's and the winner's casualty-fraction distributions are very nearly the same, independent of the attack/defense status of forces, i.e., it should explain why Eqs. (12) and (13) are approximately satisfied.

7. It would be helpful if the theory were useful for determining the winners of simulated battles, as well as for determining the casualty levels on both sides at the conclusion of simulated engagements.

In the foregoing, we have considered a primary breakpoint hypothesis and sketched three modified versions of it. Of these different versions of the breakpoint hypothesis, the one that most nearly satisfies the desiderata listed above seems to be the first modification presented. However, this modification fails to satisfy desiderata 5 and 7. Versions of the breakpoint hypothesis proposed in the foregoing that were closer to the kind normally used in war games, simulations, and maneuver control were even less satisfactory in explaining observed battle termination phenomena. Consequently, it seems that the soundness of models of combat that make essential use of breakpoint hypotheses must be considered suspect until a better theoretical understanding of the battle termination process is obtained.

Appendix A

SALTUS-FUNCTION MANIPULATIONS

The saltus-function (σ-function) and the delta-function (δ-function) enable a consistent formalism to be employed in formulae which would otherwise have to be treated by an exhaustive tabulation of cases. We present here a brief collection of some of the elementary properties of the σ-function, which is defined by

$$\sigma(x, a) = \begin{cases} 0, & \text{if } x < a \\ \frac{1}{2}, & \text{if } x = a \, . \\ 1, & \text{if } x > a \end{cases} \tag{A-1}$$

The properties of this function are intimately related to those of the Dirac δ-function, which may be nonrigorously defined by the relation

$$\sigma(x, a) = \int_{-\infty}^{x} \delta(t, a)\, dt. \tag{A-2}$$

The following relations are obvious:

$$\sigma(x, a) = \sigma(x - a, 0) \tag{A-3.1}$$

$$= 1 - \sigma(a, x) \tag{A-3.2}$$

$$= \sigma(x + y, a + y) \tag{A-3.3}$$

$$= \sigma(-a, -x) \tag{A-3.4}$$

$$\sigma(cx, ca) = \sigma(x, a)\sigma(c, 0) + \sigma(a, x)\sigma(0, c). \tag{A-4}$$

If $c \neq 0$, then

$$\sigma(cx, a) = \sigma(x, ac^{-1})\sigma(c, 0) + \sigma(ac^{-1}, x)\sigma(0, c) \tag{A-5}$$

and

$$\sigma(x, c) = \sigma(xc^{-1}, 1)\sigma(c, 0) + \sigma(1, xc^{-1})(0, c). \qquad \text{(A-6)}$$

<u>Definition</u>. If a function m satisfies the relation

$$m(x) = f(x)\sigma(a, x) + f(a)\sigma(x, a),$$

then m is said to be a mixture on f at a.

If g is any function, and m is a mixture on f at a, then

$$g(m(x)) = g(f(x))\sigma(a, x) + g(f(a))\sigma(x, a), \qquad \text{(A-7)}$$

i.e., g o m is a mixture on g o f at a, where "o" denotes functional composition.

By virtue of our convention that distribution functions are defined by the limit from the right, all upper limits in integrals must be taken as limits from the right, so that, e.g.,

$$\int_a^b f(u)\delta(u, c)\, du \equiv \int_a^{b+0} f(u)\delta(u, c)\, du$$

$$= f(c)[\sigma(b + 0, c) - \sigma(a, c)]. \qquad \text{(A-8)}$$

The most common application we shall make of relation (A-8) is for the case in which $a = 0$, $c > 0$, and $f(u) = \sigma(du, e)$. For this case (A-8) reduces to

$$\int_0^{b+0} \sigma(du, e)\delta(u, c)\, du = \sigma(dc, e)\sigma(b + 0, c). \qquad \text{(A-9)}$$

Another integral formula of some value, valid when $b \geq a$, is

$$\int_a^{b+0} f(u)\sigma(u, c)\, du = F(b + 0)\sigma(b + 0, c) - F(a)\sigma(a, c)$$

$$- F(c)[\sigma(b + 0, c) - \sigma(a, c)]$$

$$= [F(b + 0) - F(c)]\sigma(b + 0, c)$$

$$+ [F(c) - F(a)]\sigma(a, c) \qquad \text{(A-10)}$$

where F is any indefinite integral of f. In particular,

$$\int_0^{b+0} f(u)\sigma(u, c)\, du = 0, \qquad \text{for } b \leq c \text{ and F continuous from the right at b.} \qquad \text{(A-11)}$$

Appendix B

A JOSS PROGRAM FOR THE COMPUTATION OF CASUALTY-FRACTION DISTRIBUTIONS FOR EXPONENTIAL BREAK CURVES

The JOSS program for the computation of casualty-fraction distributions for exponential break curves is presented in this appendix. It is based on the discussion of casualty-fraction distributions for Example 3 (exponential breakpoints) in Appendix E. Because of JOSS program conventions, the following equivalents were established for use only in this computer routine:

Notation in Text	*Notation in Computer Routine*
λ_x	x
λ_y	y
γ	g
$\sigma(u, v)$	s(u,v)
$\gamma^{-1}\Psi(u)$	t(g,u)
$\Psi(u)$	P(g,u)
$\Psi^{-1}(u)$	p(g,u)
$P(W_x)dy(u\|W_x)$	m(x,y,g,u)
$P(f_y < u\|W_x)$	a(x,y,g,u)

THE JOSS PROGRAM

```
Type all.

1.001  Do part 10.
1.01   Demand x as "Lambda sub x".
1.02   Demand y as "Lambda sub y".
1.03   Demand g as "Gamma".
1.04   Do part 2 for u = 0(0.02)0.5(0.10)1.
1.05   To part 3.

2.01   Set A(100·u) = a(x,y,g,u).
2.02   Set B(100·u) = a(x,y,g,p(g,u)).
2.03   Set C(100·u) = a(y,x,g*(-1),u).
2.04   Set D(100·u) = a(y,x,g*(-1),P(g,u)).

3.01   Page.
3.02   Type x,y,g in form 1.
3.03   Line.
3.04   Type form 2.
3.05   Do step 3.08 for u = 0(0.02)0.5(0.10)1.
3.051  Type m(x,y,g,1) in form 4.
3.06   Page.
3.07   To step 1.01.
3.08   Type u, B(100·u), A(100·u), C(100·u), D(100·u) in form 3.

10.1   Let s(u,v) = [u<v:0;u=v:0.5;u>v:1].
10.2   Let t(g,u) = [u≤g*(-1):u;g*(-1)].
10.3   Let P(g,u) = g·t(g,u).
10.4   Let p(g,u) = g*(-1)·t(g*(-1),u).
10.5   Let r(x,y,g,u) = y·[1-exp(-(y+g·x)·t(g,u))]/(y+g·x).
10.6   Let n(x,y,g,u) = exp[-(y+g·x)]·s(1,g)·[u<1:0; 1].
10.7   Let m(x,y,g,u) = r(x,y,g,u) + n(x,y,g,u).
10.8   Let a(x,y,g,u) = m(x,y,g,u)/m(x,y,g,1).

Form 1:
Lambda sub x = ___.___; Lambda sub y = ___.___; Gamma = ___.___

Form 2:
  u     P(fx<u|Wx)   P(fy<u|Wx)   P(fx<u|Wy)   P(fy<u|Wy)

Form 3:
___.___      ___.___      ___.___      ___.___      ___.___

Form 4:
            P(W_x) = ___.______
```

Appendix C

DERIVATION OF BREAK CURVES FROM A CONTINUOUS MODEL OF DECISION BEHAVIOR

We suppose that there is a function, $X(h, f)$, that gives the probability that a side with a casualty fraction equal to f will continue to fight to a casualty fraction of $f + h$. Clearly, X must satisfy

$$X(0, f) = 1, \qquad \text{for } 0 \leq f \leq 1. \tag{C-1}$$

If $F(f)$ is the probability that the side breaks at a casualty-fraction value *less* than f, then we have the relation

$$1 - F(f + h) = X(h, f)[1 - F(f)]. \tag{C-2}$$

Subtracting $1 - F(f)$ from both sides of Eq. (C-2), dividing by h and taking the limit as h approaches zero, we find by invoking Eq. (C-1) that

$$dF/df = -\lambda(f)[F(f) - 1], \tag{C-3}$$

where we have made the notational change

$$-\lambda(f) \equiv \left.\frac{\partial X(h, f)}{\partial h}\right|_{h = 0} . \tag{C-4}$$

Integration of Eq. (C-3), subject to the initial condition that

$$F(0) = 0,$$

yields

$$F(f) = 1 - \exp\left[-\int_0^f \lambda(t)\, dt\right] \tag{C-5}$$

for the side's break curve. IF $\lambda(t)$ has a finite number of simple jump discontinuities, we should take f + 0 as the upper limit of integration in Eq. (C-5). Because of the convention that $F(1 + 0) = 1$, we must either have

$$\exp\left[-\int_0^{1+0} \lambda(t)\, dt\right] = 0 \qquad \text{(C-6)}$$

or introduce an *ad hoc* term to give $F(1 + 0) = 1$.

Appendix D

COLLECTION OF BASIC FORMULAE RESULTS

$$P(W_y) = \int_0^1 F_x(\Psi(u))\ dF_y(u) \qquad \text{(D-1)}$$

$$= \int_0^1 [1 - F_y(\Psi^{-1}(v))]\ dF_x(v) \qquad \text{(D-2)}$$

$$P(W_x) = \int_0^1 F_y(\Psi^{-1}(v)\ dF_x(v)) \qquad \text{(D-3)}$$

$$= \int_0^1 [1 - F_x(\Psi(u))]\ dF_y(u) \qquad \text{(D-4)}$$

$$dD_y(u|W_x) = \frac{1}{P(W_x)} [1 - F_x(\Psi(u))]\ dF_y(u) = P(L_y = u|W_x) \qquad \text{(D-5)}$$

$$P(f_y < q|W_x) = D_y(q|W_x) = \Delta_{yx}(q) \qquad \text{(D-6)}$$

$$P(f_x < s|W_x) = D_y(\Psi^{-1}(s)|W_x) = P(f_y < \Psi^{-1}(s)|W_x) = \Delta_{xx}(s) \qquad \text{(D-7)}$$

$$dD_x(v|W_y) = \frac{1}{P(W_y)} [1 - F_y(\Psi^{-1}(v))]\ dF_x(v) = P(L_x = v|W_y) \qquad \text{(D-8)}$$

$$P(f_x < s|W_y) = D_x(s|W_y) = \Delta_{xy}(s) \qquad \text{(D-9)}$$

$$P(f_y < q|W_y) = D_x(\Psi(q)|W_y) = P(f_x < \Psi(q)|W_y) = \Delta_{yy}(q) \qquad \text{(D-10)}$$

Appendix E

ILLUSTRATIVE EXAMPLES

The breakpoint model (Hypotheses A, B, and C) is illustrated below by working out some simplified examples. The purpose here is to make sure that the model gives results in these simpler cases that agree with the intuitive implications of the assumptions made.

EXAMPLE 1: FIXED BREAKPOINTS

Suppose that side z is sure to break off the attack at a casualty-fraction value of k_z, but not before. Then we can write the break curves as

$$F_z(u) = \sigma(u, k_z),$$

where σ is the step, or saltus, function defined by

$$\sigma(x, a) = \begin{cases} 0, & \text{for } x < a \\ \frac{1}{2}, & \text{for } x = a \\ 1, & \text{for } x > a \end{cases}$$

and $0 < k_z \leq 1$. Appendix A develops some of the calculus of such functions. However, for Example 1 it is perhaps as convenient to proceed on the basis of informal considerations.

We proceed to find

$$P(W_y) = \int_0^1 F_x(\Psi(u))\, dF_y(u)$$

$$= \int_0^{1+0} \sigma(\Psi(u), k_x)\delta(u, k_y)\, du$$

$$= \sigma(\Psi(k_y), k_x),$$

where the last line follows from formula (A-8) of Appendix A, since $0 < k_y \le 1$.

Also

$$P(W_x) = \int_0^1 F_y(\Psi^{-1}(v))\, dF_x(v)$$

$$= \int_0^{1+0} \sigma(\Psi^{-1}(v),\, k_y)\delta(v,\, k_x)\, dv$$

$$= \sigma(\Psi^{-1}(k_x),\, k_y)$$

$$= \sigma(k_x,\, \Psi(k_y)),$$

which by Eq. (A-3.2) is equivalent to $1 - P(W_y)$, as it should be.

We proceed to form

$$D_y(q|W_x)P(W_x) = \int_0^q [1 - F_x(\Psi(u))]\, dF_y(u)$$

$$= \int_0^{q+0} [1 - \sigma(\Psi(u),\, k_x)]\delta(u,\, k_y)\, du$$

$$= [1 - \sigma(\Psi(k_y),\, k_x)]\sigma(q + 0,\, k_y).$$

Now, in order that $P(W_x) > 0$, we must have $k_x > \Psi(k_y)$, in which case the first factor on the right-hand side will be unity. Hence we may write

$$D_y(q|W_x) = \frac{1}{P(W_x)}\, \sigma(q + 0,\, k_y) = \sigma(q + 0,\, k_y).$$

Accordingly, we have

$$P(f_y < q|W_x) = D_y(q|W_x) = \sigma(q + 0,\ k_y)$$

and

$$\begin{aligned} P(f_x < s|W_x) &= D_y(\Psi^{-1}(s)|W_x) \\ &= \sigma(\Psi^{-1}(s) + 0,\ k_y) \\ &= \sigma(s + 0,\ \Psi(k_y)). \end{aligned}$$

That is, $f_y = k_y$ and $f_x = \Psi(k_y)$, with probability one. But that is exactly what we expect in this example when x wins.

Similarly,

$$\begin{aligned} P(W_y)\ D_x(s|W_y) &= \int_0^s [1 - F_y(\Psi^{-1}(v))]\ dF_x(v) \\ &= \int_0^{s+0} [1 - \sigma(\Psi^{-1}(v),\ k_y)]\delta(v,\ k_x)dv \\ &= [1 - \sigma(\Psi^{-1}(k_x),\ k_y)]\sigma(s + 0,\ k_x). \end{aligned}$$

For y to win, we must have $k_x \leq \Psi(k_y)$, in which case the first factor on the right-hand side will be non-zero, and we may write

$$D_x(s|W_y) = \sigma(s + 0,\ k_x).$$

Accordingly, we have

$$P(f_x < s|W_y) = D_x(s|W_y) = \sigma(s + 0,\ k_x)$$

and

$$P(f_y < q \mid W_y) = D_x(\Psi(q) \mid W_y)$$

$$= \sigma(\Psi(q) + 0,\ k_x) = \sigma(q + 0,\ \Psi^{-1}(k_x)).$$

That is, $f_x = k_x$ and $f_y = \Psi^{-1}(k_x)$, with probability one. But this is exactly what we expect in this example when y wins.

EXAMPLE 2: UNIFORM BREAKPOINTS

Suppose that each side is equally likely to break off the engagement at any point, i.e., we suppose that

$$F_z(u) = u, \qquad \text{for } 0 \le u \le 1.$$

We will also suppose that $\varphi(u) = \gamma u$, so that $\Psi(u) = \gamma u\sigma(\gamma^{-1}, u) + \sigma(u, \gamma^{-1})$ and $\Psi^{-1}(u) = \gamma^{-1}u\sigma(\gamma, u) + \sigma(u, \gamma)$ for $0 \le u \le 1$. Here, γ is some positive constant of proportionality.

We proceed to find

$$P(W_y) = \int_0^1 F_x(\Psi(u))\ dF_y(u)$$

$$= \tfrac{1}{2}\gamma\sigma(1, \gamma) + (\tfrac{1}{2}\gamma^{-1} + 1 - \gamma^{-1})\sigma(\gamma, 1)$$

$$= \tfrac{1}{2}\gamma\sigma(1, \gamma) + (1 - \tfrac{1}{2}\gamma^{-1})\sigma(\gamma, 1).$$

Also

$$P(W_x) = \int_0^1 F_y(\Psi^{-1}(v))\ dF_x(v)$$

$$= \int_0^1 \Psi^{-1}(v)\ dv$$

$$= \tfrac{1}{2}\gamma^{-1}\sigma(1, \gamma^{-1}) + (1 - \tfrac{1}{2}\gamma)\sigma(\gamma^{-1}, 1).$$

We form

$$P(W_x)\ D_y(q|W_x) = \int_0^q [1 - F_x(\Psi(u))]\ dF_y(u)$$

$$= \int_0^q [1 - \Psi(u)]\ du$$

$$= \int_0^q [1 - \gamma u \sigma(\gamma^{-1}, u) - \sigma(u, \gamma^{-1})]\ du$$

$$= (q - \tfrac{1}{2}\gamma q^2)\sigma(\gamma^{-1}, q) + \tfrac{1}{2}\gamma^{-1}\sigma(q, \gamma^{-1}).$$

The right-hand side reduces to the previous expression for $P(W_x)$ when $q = 1$, as it should.*

Similarly, we form

$$P(W_y)\ D_x(s|W_y) = \int_0^s [1 - F_y(\Psi^{-1}(v))]\ dF_x(v)$$

$$= \int_0^s [1 - \Psi^{-1}(v)]\ dv$$

$$= (s - \tfrac{1}{2}\gamma^{-1}s^2)\sigma(\gamma, s) + \tfrac{1}{2}\gamma\sigma(s, \gamma),$$

which reduces to the previous expression for $P(W_y)$ when $q = 1$, as it should. When $\gamma = 1$, $\Psi(u) = \varphi(u) = u$ for $0 \le u \le 1$, and $P(W_x) = P(W_y) = \frac{1}{2}$,

---

*In this example, as in the preceding one, it is not necessary to use the formal manipulations of the saltus functions presented in Appendix A. It is easier to use instead the definition of the saltus function directly, keeping the various cases in mind as one proceeds.

and we have $P(f_y < q|W_x) = P(f_x < q|Wx) = P(f_y < q|Wy) = P(f_x < q|W_y) = 2q - q^2$.

Example 2.1 ($\gamma \geq 1$)

When $\gamma > 1$, we have

$$P(W_y) = 1 - \tfrac{1}{2}\gamma^{-1} = 1 - (2\gamma)^{-1}$$

$$P(W_x) = \tfrac{1}{2}\gamma^{-1} = (2\gamma)^{-1}$$

$$\begin{aligned} P(f_y < q|W_x) &= D_y(q|W_x) \\ &= \gamma(2q - \gamma q^2)\sigma(\gamma^{-1}, q) + \sigma(q, \gamma^{-1}) \\ &= [2(q\gamma) - (q\gamma)^2]\sigma(\gamma^{-1}, q) + \sigma(q, \gamma^{-1}) \end{aligned}$$

$$\begin{aligned} P(f_x < s|W_x) = D_y(\Psi^{-1}(s)|W_x) &= \gamma(2(\gamma^{-1}s) - \gamma(\gamma^{-2}s^2))\sigma(\gamma^{-1}(\gamma^{-1}s)) \\ &\quad + \sigma((\gamma^{-1}s), \gamma^{-1}), \end{aligned}$$

and by Eq. (A-4) in Appendix A this reduces to

$$\begin{aligned} &= \gamma(2s\gamma^{-1} - s^2\gamma^{-1})\sigma(1, s) \\ &\quad + \sigma(s, 1) \\ &= (2s - s^2)\sigma(1, s) + \sigma(s, 1) \\ &= 2s - s^2. \end{aligned}$$

It is interesting to note that when $\gamma > 1$, the conditional distribution of attacker casualty fraction given an attacker win is independent of the proportionality factor, γ.

In similar fashion, we find

$$P(f_x < s|W_y) = D_x(s|W_y)$$

$$= \frac{(s - \frac{1}{2}\gamma^{-1}s^2)}{1 - \frac{1}{2}\gamma^{-1}}$$

$$= \frac{2\gamma s - s^2}{2\gamma - 1}.$$

And also

$$P(f_y < q|W_y) = D_x(\Psi(q)|W_y)$$

$$= (2\gamma - 1)^{-1}\{2\gamma\Psi(q) - \Psi^2(q)\}$$

$$= (2\gamma - 1)^{-1}\{[2\gamma\gamma q - (\gamma q)^2]\sigma(\gamma^{-1}, q)$$

$$+ [2\gamma - 1]\sigma(q, \gamma^{-1})\}$$

$$= \frac{2\gamma(\gamma q) - (\gamma q)^2}{2\gamma - 1}\sigma(\gamma^{-1}, q) + \sigma(q, \gamma^{-1}).$$

Note that all of these formulae will reduce to those for the case in which $\gamma = 1$ upon setting $\gamma = 1$.

<u>Example 2.2 ($\gamma \leq 1$)</u>

When $\gamma < 1$,

$$P(W_y) = \tfrac{1}{2}\gamma$$

$$P(W_x) = 1 - \tfrac{1}{2}\gamma$$

$$P(f_y < q|W_x) = D_y(q|W_x) = \frac{(q - \frac{1}{2}\gamma q^2)}{1 - \frac{1}{2}\gamma}$$

$$= \frac{2q - \gamma q^2}{2 - \gamma}$$

$$P(f_x < s|W_x) = D_y(\Psi^{-1}(s)|W_x)$$

$$= \frac{2\Psi^{-1}(s) - \gamma(\Psi^{-1}(s))^2}{2 - \gamma}$$

$$= \frac{2\gamma^{-1}s - \gamma(\gamma^{-1}s)^2}{2 - \gamma}\,\sigma(\gamma,\ s) + \sigma(s,\ \gamma).$$

Likewise,

$$P(f_x < s|W_y) = D_x(s|W_y)$$

$$= \frac{(s - \tfrac{1}{2}\gamma^{-1}s^2)\sigma(\gamma,\ s) + \tfrac{1}{2}\gamma\sigma(s,\ \gamma)}{\tfrac{1}{2}\gamma}$$

$$= [2(\gamma^{-1}s) - (\gamma^{-1}s)^2]\sigma(\gamma,\ s) + \sigma(s,\ \gamma)$$

$$P(f_y < q|W_y) = D_x(\Psi(q)|W_y)$$

$$= [2(\gamma^{-1}\Psi(q)) - (\gamma^{-1}\Psi(q))^2]\sigma(\gamma,\ \Psi(q)) + \sigma(\Psi(q),\ \gamma)$$

$$= [2(\gamma^{-1}\gamma q) - (\gamma^{-1}\gamma q)^2]\sigma(\gamma,\ \gamma q) + \sigma(\gamma q,\ \gamma)$$

$$= [2q - q^2]\sigma(1,\ q) + \sigma(q,\ 1)$$

$$= 2q - q^2.$$

And, symmetrically related to a previous result, the conditional distribution of defender casualty fractions when the defender wins is independent of the proportionality factor, γ. As is true for the case in which $\gamma > 1$, these results will reduce to those for the case in which $\gamma = 1$ simply by taking $\gamma = 1$.

EXAMPLE 3: EXPONENTIAL BREAKPOINTS

Suppose that each side's break curve is given by an exponential

function

$$F_z(u) = 1 - e^{-\lambda_z u} + e^{-\lambda_z}\sigma(u, 1), \qquad 0 \le u \le 1,$$

$$dF_z(u) = [\lambda_z e^{-\lambda_z u} + e^{-\lambda_z}\delta(u, 1)]\ du,$$

where the last term is inserted to make $F_z(1 + 0) = 1$, and so to cure the defect of the distribution

$$1 - e^{-\lambda_z u}$$

at $u = 1$.*

We shall also suppose that

$$\varphi(u) = \gamma u$$

as we did for Example 2.

We proceed to find

$$P(W_y) = \int_0^{1+0} F_x(\Psi(u))\ dF_y(u)$$

$$= \int_0^{1+0} [1 - e^{-\lambda_x \Psi(u)} + e^{-\lambda_x}\sigma(\Psi(u), 1)]$$

$$[\lambda_y e^{-\lambda_y u}\ du + e^{-\lambda_y}\delta(u, 1)\ du].$$

---

*It also, somewhat regrettably, has the effect of attaching a finite probability to the event "force z fights to complete annihilation," by way of the δ-function term in dF_z. One way to avoid this would have been to simply rescale F_z by a factor, e.g., set $F_z = (1 - e^{-\lambda z u})/(1 - e^{-\lambda z})$. This apparently simpler and more natural technique will not be developed here.

We proceed by first taking $\gamma \le 1$, and finding* the value of

$$P(W_x)\, D_y(q|W_x) = \int_0^{q+0} [1 - F_x(\Psi(u))]\, dF_y(u)$$

$$= \int_0^{q+0} [e^{-\lambda_x \gamma u} - e^{-\lambda_x}\sigma(\gamma u, 1)]$$

$$[\lambda_y e^{-\lambda_y u} + e^{-\lambda_y}\Delta(u, 1)]\, du$$

$$= \frac{\lambda_y}{\lambda_y + \gamma\lambda_x}\left[1 - e^{-(\lambda_y+\gamma\lambda_x)\gamma^{-1}\Psi(q)}\right]$$

$$+ e^{-(\lambda_y+\gamma\lambda_x)}\sigma(1, \gamma)\sigma(q + 0, 1).$$

The value of $P(W_x)$ can be found by taking $q = 1$ in the above, since $D_y(1|W_x)$ must equal unity. Hence $P(f_y < q|W_x) = D_y(q|W_x)$ is given by dividing the right-hand side of the last equality by the value obtained by setting $q = 1$ in the same expression. Then we can find

$$P(f_x < s|W_x) = D_y(\Psi^{-1}(s)|W_x),$$

where $\Psi^{-1}(s) = \gamma^{-1}s\sigma(\gamma, s) + \sigma(s, \gamma)$, because Ψ^{-1} is a mixture on $\gamma^{-1}s$ at γ. By fact (A-7) in Appendix A, and since $D_y(1|W_x) = 1$, we obtain $P(f_x < s|W_x) = P(f_y < \gamma^{-1}s|W_x)\sigma(\gamma, s) + \sigma(s, \gamma)$.

When side y wins, we find similarly

$$P(W_y)\, D_x(s|W_y) = \frac{\gamma\lambda_x}{\lambda_y + \gamma\lambda_x}\left[1 - e^{-(\lambda_y+\gamma\lambda_x)\Psi^{-1}(s)}\right]$$

$$+ e^{-(\lambda_x+\gamma^{-1}\lambda_y)}\sigma(1, \gamma^{-1})\sigma(s + 0, 1),$$

---

*Throughout this example, frequent use is made of the results in Appendix A. Particularly heavy use is made of formula (A-9). The details are straightforward, though somewhat tedious.

and

$$P(f_x < s|W_y) = D_x(s|W_y)$$

can be found by the same method used above to obtain $P(f_x < s|W_x)$. Likewise, we obtain

$$P(f_y < q|W_y) = D_x(\Psi(q)|W_y).$$

We work out the results for $\gamma \geq 1$ in analogous fashion. Whether or not $\gamma < 1$, $\gamma = 1$, or $\gamma > 1$, we have in any event

$$P(W_x)\ D_y(q|W_x) = \frac{\lambda_y}{\lambda_y + \gamma\lambda_x}\left(1 - e^{-(\lambda_y+\gamma\lambda_x)\gamma^{-1}\Psi(q)}\right)$$

$$+ e^{-(\lambda_y+\gamma\lambda_x)}\sigma(1,\ \gamma)\sigma(q + 0,\ 1) \qquad \text{(E-1)}$$

and a similar formula for $P(W_y)\ D_x(q|W_y)$ obtainable from the one for $P(W_x)\ D_y(q|W_x)$ by the mapping* $x \to y$, $y \to x$, $\gamma \to \gamma^{-1}$, $\Psi \to \Psi^{-1}$, and $\Psi^{-1} \to \Psi$. Here $\Psi(q) = \gamma q\sigma(\gamma^{-1},\ q) + \sigma(q,\ \gamma^{-1})$, so by relation (D-6) of Appendix D,

$$P(f_y < q|W_x) = D_y(q|W_x)$$

$$= \frac{\frac{\lambda_y}{\lambda_y + \gamma\lambda_x}\left(1 - e^{-(\lambda_y+\gamma\lambda_x)\gamma^{-1}\Psi(q)}\right) + e^{-(\lambda_y+\gamma\lambda_x)}\sigma(1,\ \gamma)\sigma(q + 0,\ 1)}{\frac{\lambda_y}{\lambda_y + \gamma\lambda_x}\left(1 - e^{-(\lambda_y+\gamma\lambda_x)\gamma^{-1}\Psi(1)}\right) + e^{-(\lambda_y+\gamma\lambda_x)}\sigma(1,\ \gamma)}$$

(E-2)

---

*This interchange of symbols in any expression, formula, etc., will be called the "usual transposition." The object to which the usual transposition is applied will be called the "primal object." The result of applying the usual transposition to the primal will be called the "dual."

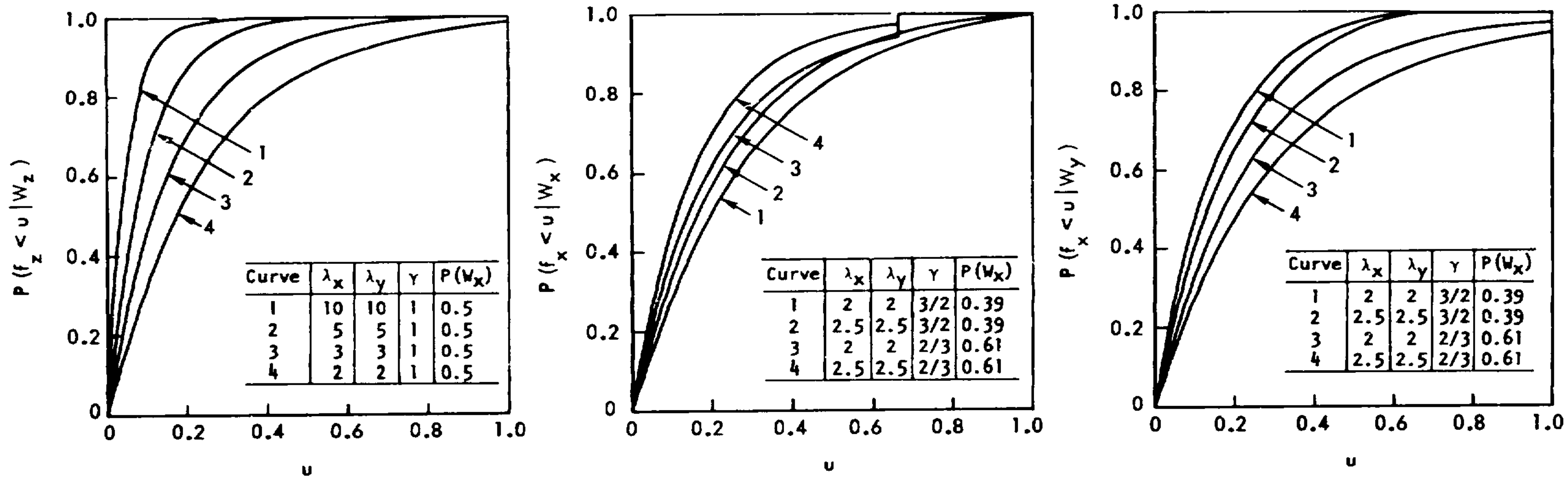
with a similar formula for $P(f_x < q|W_y)$ dual to (E-2) obtainable by the usual transposition. From relation (D-7) of Appendix D we obtain

$$P(f_x < s|W_x) = D_y(\Psi^{-1}(s)|W_x) = P(f_y < \Psi^{-1}(s)|W_x), \qquad \text{(E-3)}$$

where $\Psi^{-1}(s) = \gamma^{-1}s\sigma(\gamma, s) + \sigma(s, \gamma)$, and so is a mixture on $\gamma^{-1}s$. Then by Eq. (A-7) we may write

$$P(f_x < s|W_x) = P(f_y < \gamma^{-1}s|W_x)\sigma(\gamma, s) + \sigma(s, \gamma), \qquad \text{(E-4)}$$

with an analogous dual formula obtainable from this one by the usual transposition. A JOSS program for the computation of theoretical casualty-fraction distributions for this example is included as Appendix B. This program was used to generate the following tables of values (Tables E-1 through E-5). Some of these distributions are illustrated in Fig. E-1.



(a) Given a win by either side

(b) Given that the attacker wins

(c) Given that the defender wins

Fig.E-1 —Theoretical casualty-fraction distributions

Table E-1

THEORETICAL CASUALTY-FRACTION DISTRIBUTION[a]

($\gamma = 1$, $\lambda_x = \lambda_y = \lambda$)

u	$P(f_z<u\|W_z)$			
	λ=10	λ=5	λ=3	λ=2
.000	.0000	.0000	.0000	.0000
.020	.3297	.1813	.1131	.0769
.040	.5507	.3297	.2134	.1479
.060	.6988	.4512	.3023	.2134
.080	.7981	.5507	.3812	.2739
.100	.8647	.6321	.4512	.3297
.120	.9093	.6988	.5132	.3812
.140	.9392	.7534	.5683	.4288
.160	.9592	.7981	.6171	.4727
.180	.9727	.8347	.6604	.5132
.200		.8647	.6988	.5507
.220		.8892	.7329	.5852
.240		.9093	.7631	.6171
.260		.9257	.7899	.6465
.280		.9392	.8136	.6737
.300		.9502	.8347	.6988
.320		.9592	.8534	.7220
.340			.8700	.7433
.360			.8847	.7631
.380			.8977	.7813
.400			.9093	.7981
.420			.9195	.8136
.440			.9286	.8280
.460			.9367	.8412
.480			.9439	.8534
.500			.9502	.8647
.600			.9727	.9093
.700			.9850	.9392
.800				.9592
.900				.9727
1.000				1.0000

[a][$P(f_x<u|W_x) = P(f_y<u|W_x) = P(f_x<u|W_y) = P(f_y<u|W_y)$, $P(W_x) = P(W_y) = 1/2$].

Table E-2

THEORETICAL CASUALTY-FRACTION DISTRIBUTION[a]

(λ_x = 2.000; λ_y = 2.000; γ = 1.500)

u	$P(f_x<u\|W_x)$	$P(f_y<u\|W_x)$	$P(f_x<u\|W_y)$	$P(f_y<u\|W_y)$
.000	.0000	.0000	.0000	.0000
.020	.0669	.0987	.0630	.0930
.040	.1294	.1880	.1219	.1771
.060	.1880	.2688	.1771	.2532
.080	.2427	.3419	.2286	.3220
.100	.2940	.4080	.2769	.3843
.120	.3419	.4679	.3220	.4407
.140	.3867	.5220	.3642	.4917
.160	.4286	.5710	.4038	.5379
.180	.4679	.6154	.4407	.5796
.200	.5046	.6555	.4753	.6174
.220	.5389	.6918	.5076	.6516
.240	.5710	.7247	.5379	.6826
.260	.6011	.7544	.5662	.7106
.280	.6292	.7813	.5927	.7359
.300	.6555	.8036	.6174	.7588
.320	.6801	.8276	.6406	.7796
.340	.7031	.8476	.6623	.7983
.360	.7247	.8656	.6826	.8153
.380	.7448	.8819	.7015	.8307
.400	.7636	.8967	.7193	.8446
.420	.7813	.9100	.7359	.8572
.440	.7978	.9221	.7514	.8685
.460	.8132	.9330	.7660	.8788
.480	.8276	.9429	.7796	.8882
.500	.8411	.9519	.7923	.8966
.600	.8967	.9854	.8446	.9281
.700	.9364	1.0000	.8821	1.0000
.800	.9649	1.0000	.9089	1.0000
.900	.9854	1.0000	.9281	1.0000
1.000	1.0000	1.0000	1.0000	1.0000

[a]$P(W_x)$ = 0.385730402.

Table E-3

THEORETICAL CASUALTY-FRACTION DISTRIBUTION[a]

(λ_x = 2.500; λ_y = 2.500; γ = 1.500)

u	$P(f_x<u\|W_x)$	$P(f_y<u\|W_x)$	$P(f_x<u\|W_y)$	$P(f_y<u\|W_y)$
.000	.0000	.0000	.0000	.0000
.020	.0812	.1194	.0791	.1163
.040	.1559	.2247	.1519	.2189
.060	.2247	.3176	.2189	.3095
.080	.2879	.3997	.2806	.3894
.100	.3461	.4721	.3373	.4600
.120	.3997	.5359	.3894	.5222
.140	.4489	.5923	.4374	.5772
.160	.4942	.6421	.4816	.6257
.180	.5359	.6860	.5222	.6684
.200	.5743	.7247	.5596	.7062
.220	.6096	.7589	.5940	.7395
.240	.6421	.7891	.6257	.7689
.260	.6720	.8157	.6548	.7949
.280	.6994	.8392	.6816	.8178
.300	.7247	.8600	.7062	.8380
.320	.7480	.8783	.7289	.8558
.340	.7694	.8944	.7497	.8716
.360	.7891	.9087	.7689	.8854
.380	.8072	.9213	.7866	.8977
.400	.8239	.9324	.8028	.9085
.420	.8392	.9422	.8178	.9181
.440	.8534	.9508	.8315	.9265
.460	.8663	.9584	.8442	.9339
.480	.8783	.9652	.8558	.9405
.500	.8893	.9711	.8665	.9463
.600	.9324	.9919	.9085	.9665
.700	.9608	1.0000	.9362	1.0000
.800	.9795	1.0000	.9545	1.0000
.900	.9919	1.0000	.9665	1.0000
1.000	1.0000	1.0000	1.0000	1.0000

[a] $P(W_x)$ = 0.393798459.

Table E-4

THEORETICAL CASUALTY-FRACTION DISTRIBUTION[a]

(λ_x = 2.000; λ_y = 2.000; γ = 0.667)

u	$P(f_x<u\|W_x)$	$P(f_y<u\|W_x)$	$P(f_x<u\|W_y)$	$P(f_y<u\|W_y)$
.000	.0000	.0000	.0000	.0000
.020	.0930	.0630	.0987	.0669
.040	.1771	.1219	.1880	.1294
.060	.2532	.1771	.2688	.1880
.080	.3220	.2286	.3419	.2427
.100	.3843	.2769	.4080	.2940
.120	.4407	.3220	.4679	.3419
.140	.4917	.3642	.5220	.3867
.160	.5379	.4038	.5710	.4286
.180	.5796	.4407	.6154	.4679
.200	.6174	.4753	.6555	.5046
.220	.6516	.5076	.6918	.5389
.240	.6826	.5379	.7247	.5710
.260	.7106	.5662	.7544	.6011
.280	.7359	.5927	.7813	.6292
.300	.7588	.6174	.8056	.6555
.320	.7796	.6406	.8276	.6801
.340	.7983	.6623	.8476	.7031
.360	.8153	.6826	.8656	.7247
.380	.8307	.7015	.8819	.7448
.400	.8446	.7193	.8967	.7636
.420	.8572	.7359	.9100	.7813
.440	.8685	.7514	.9221	.7978
.460	.8788	.7660	.9330	.8132
.480	.8882	.7796	.9429	.8276
.500	.8966	.7923	.9519	.8411
.600	.9281	.8446	.9854	.8967
.700	1.0000	.8821	1.0000	.9364
.800	1.0000	.9089	1.0000	.9649
.900	1.0000	.9281	1.0000	.9854
1.000	1.0000	1.0000	1.0000	1.0000

[a]$P(W_x)$ = 0.614269595.

Table E-5

THEORETICAL CASUALTY-FRACTION DISTRIBUTION[a]

(λ_x = 2.500; λ_y = 2.500; γ = 0.667)

u	$P(f_x<u\|W_x)$	$P(f_y<u\|W_x)$	$P(f_x<u\|W_y)$	$P(f_y<u\|W_y)$
.000	.0000	.0000	.0000	.0000
.020	.1163	.0791	.1194	.0812
.040	.2189	.1519	.2247	.1559
.060	.3095	.2189	.3176	.2247
.080	.3894	.2806	.3997	.2879
.100	.4600	.3373	.4721	.3461
.120	.5222	.3894	.5359	.3997
.140	.5772	.4374	.5923	.4489
.160	.6257	.4816	.6421	.4942
.180	.6684	.5222	.6860	.5359
.200	.7062	.5596	.7247	.5743
.220	.7395	.5940	.7589	.6096
.240	.7689	.6257	.7891	.6421
.260	.7949	.6548	.8157	.6720
.280	.8178	.6816	.8392	.6994
.300	.8380	.7062	.8600	.7247
.320	.8558	.7289	.8783	.7480
.340	.8716	.7497	.8944	.7694
.360	.8854	.7689	.9087	.7891
.380	.8977	.7866	.9213	.8072
.400	.9085	.8028	.9324	.8239
.420	.9181	.8178	.9422	.8392
.440	.9265	.8315	.9508	.8534
.460	.9339	.8442	.9584	.8663
.480	.9405	.8558	.9652	.8783
.500	.9463	.8665	.9711	.8893
.600	.9665	.9085	.9919	.9324
.700	1.0000	.9362	1.0000	.9608
.800	1.0000	.9545	1.0000	.9795
.900	1.0000	.9665	1.0000	.9919
1.000	1.0000	1.0000	1.0000	1.0000

[a] $P(W_x)$ = 0.60620154.

Appendix F

A REVIEW OF SOME CASUALTY-FRACTION DATA

Some of the available casualty-fraction distribution data are exhibited in Fig. F-1. Such backup data for this figure as have not been previously presented are given in Tables F-1 through F-3. Several additional casualty-fraction distributions can be found in Refs. 17 and 18, but these distributions give the casualty-fraction distributions for one side only, and so do not suffice for the quantitative test of breakpoint hypotheses. It is interesting to note, though, that the casualty-fraction distributions given in those references illustrate the same general form as those in Fig. F-1, a shape that Robert J. Best of the Research Analysis Corporation has called "quasi-exponential." Best found the same qualitative shape in distributions of daily casualty incidence for units from rifle companies to Army groups, although the quantitative characteristics of these distributions are different for units of markedly different sizes.

Some additional data from Ref. 4 are shown in Fig. F-2. The three types of breaks displayed in this figure are defined as follows:

Type I = a sequence of attack, to reorganization, to renewal of the attack.

Type II = a change from attack to defense.

Type III = a change from defense to withdrawal.

Clark's Type I and Type II breaks perhaps should be combined and treated as a single category, as we have done in (a) of Fig. 9. There seems to be no way to foresee whether an attempted reorganization would permit a renewal of the assault. Even if a renewed assault is planned by the commander, the break may end up being of Type II because, for example, an enemy counterattack spoils the anticipated renewal of the offensive.

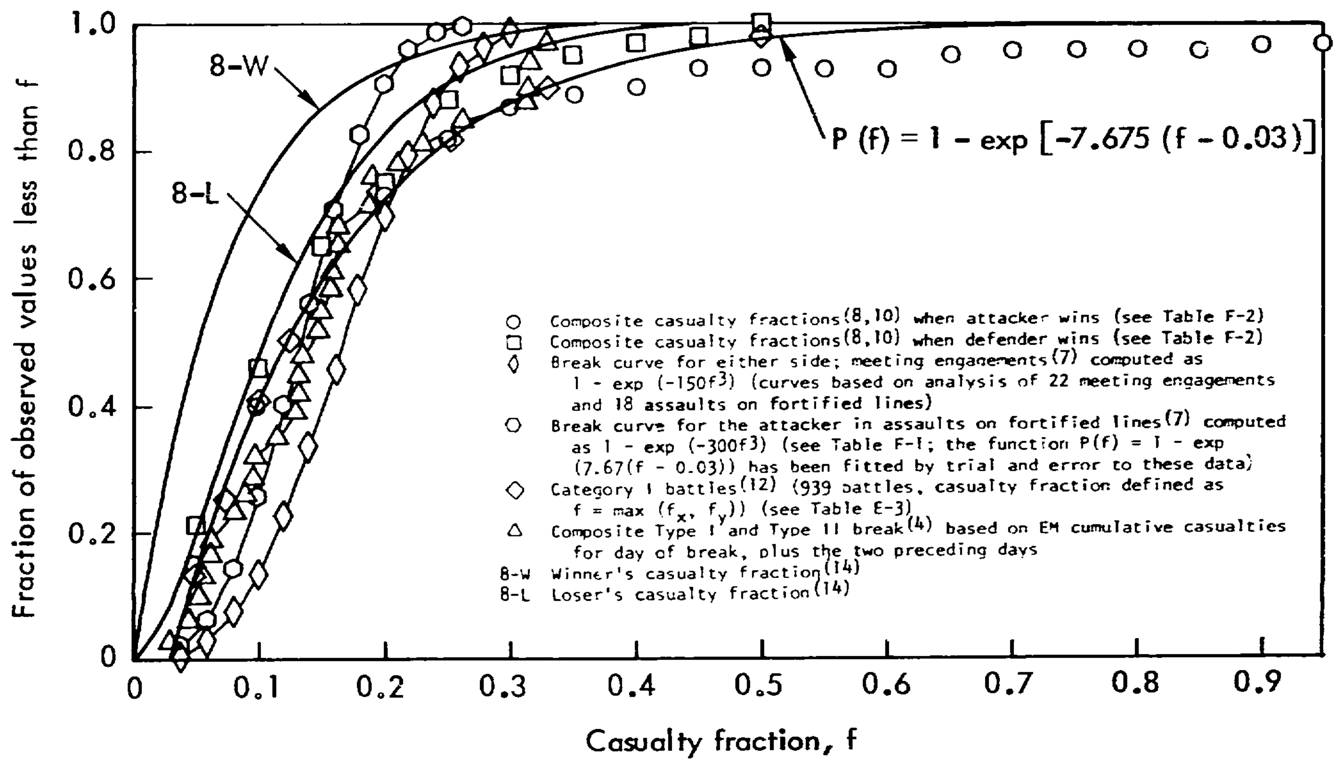


Fig. F-1 —Collection of empirical casualty-fraction distributions

Table F-1

EMPIRICAL DISTRIBUTION OF CASUALTY-FRACTION VALUES
(Extracted from Willard[(12)] for Category I battles[a])

Casualty Fraction, f	Fraction of Battles with Upper Casualty Fraction[b] Less Than f
0.05	0.13
0.075	0.25
0.10	0.41
0.125	0.50
0.20	0.73
0.25	0.82
0.33	0.90
0.50	0.98

[a]Category I battles are those characterized by G. Bodart's *Kriegs-Lexicon* as *treffen*, *gefecht*, and *schlacht*.

[b]Upper casualty fraction = max (f_x, f_y).

Table F-2

EMPIRICAL DISTRIBUTION OF CASUALTY-FRACTION VALUES[a]
(Composite from Refs. 8 and 10)

Range of Casualty-Fraction Values	Attacker Wins (W_x)			Defender Wins (W_y)			Either Side Wins		
	No. Battles	Cum. No. Battles	Cum. %	No. Battles	Cum. No. Battles	Cum. %	No. Battles	Cum. No. Battles	Cum. %
0.00-0.05	28	28	15	33	33	21	61	61	18
0.05-0.10	49	77	41	40	73	46	89	150	43
0.10-0.15	31	108	57	29	102	65	60	210	61
0.15-0.20	30	138	73	15	117	74	45	255	74
0.20-0.25	17	155	82	22	139	88	39	294	85
0.25-0.30	8	163	87	7	146	92	15	309	89
0.30-0.35	5	168	89	4	150	95	9	318	92
0.35-0.40	1	169	90	4	154	97	5	323	93
0.40-0.45	5	174	93	1	155	98	6	329	95
0.45-0.50	1	175	93	3	158	100	4	333	96
0.50-0.55	0	175	93	0	158	100	0	333	96
0.55-0.60	0	175	93	..	..	..	0	333	96
0.60-0.65	3	178	95	..	..	..	3	336	97
0.65-0.70	2	180	96	..	..	..	2	338	98
0.70-0.75	0	180	96	..	..	..	0	338	98
0.75-0.80	1	181	96	..	..	..	1	339	98
0.80-0.85	0	181	96	..	..	..	0	339	98
0.85-0.90	1	182	97	..	..	..	1	340	98
0.90-0.95	1	183	97	..	..	..	1	341	99
0.95-1.00	5	188	100	..	..	..	5	346	100

[a]Number of battles = number of battles in which one side or another experienced a casualty fraction in the appropriate range. Total number

Table F-3

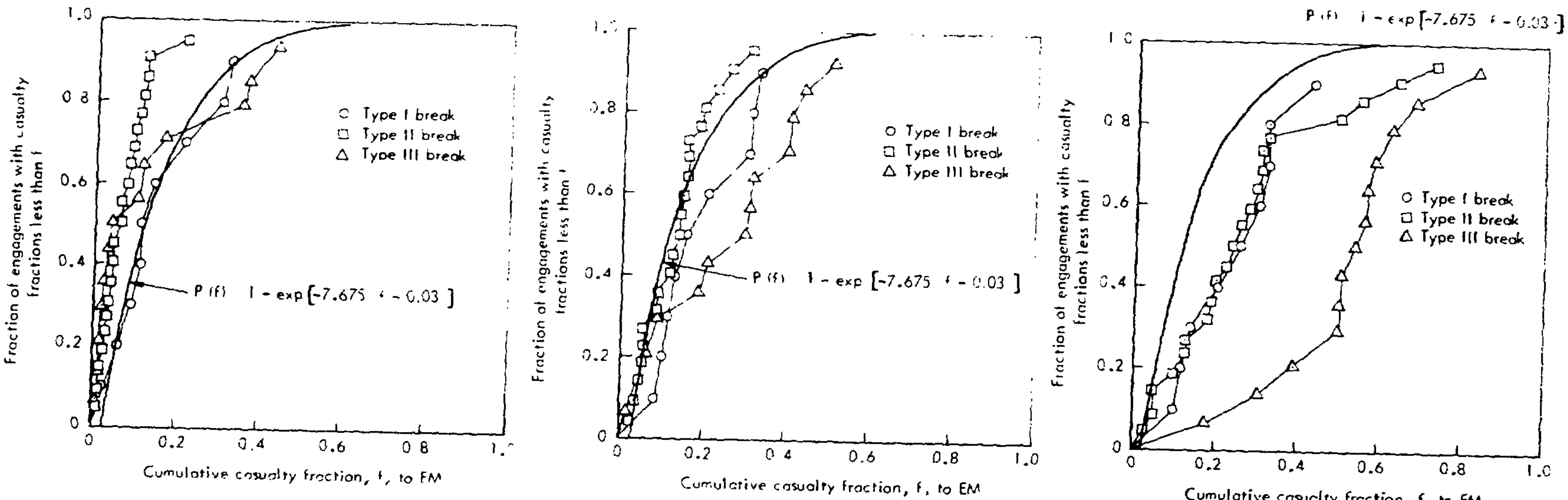
EMPIRICAL DISTRIBUTION OF CASUALTY-FRACTION VALUES

(Composite of Type I and Type II breaks,[a] EM cumulative casualties for day of break plus the two preceding days[(4)])

Casualty Fraction	Cumulative Percent[b]
0.030	3
0.045	6
0.048	10
0.053	13
0.064	16
0.064	19
0.086	23
0.091	26
0.097	29
0.097	32
0.116	35
0.128	39
0.130	42
0.132	45
0.137	48
0.141	52
0.147	55
0.156	58
0.159	61
0.162	65
0.164	68
0.181	71
0.194	74
0.211	78
0.229	81
0.262	84
0.311	87
0.313	90
0.314	94
0.326	97

[a]Type I break is defined as a change from attack to reorganization and then return to attack. Type II break is defined as a change from attack to defense.

[b]There were 30 battles in the data sample. The cumulative percentage value associated with the i^{th} battle was taken as $100 \cdot i/31$.



(a) For casualties suffered by EM on day of break

(b) For casualties suffered by EM on day of break plus two preceding days

(c) For casualties suffered by EM from start of campaign to day of break

Fig. F-2 — Empirical break curves for battalion casualties (4)

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REFERENCES

1. *Maneuver Control*, Department of the Army Field Manual, FM 105-5, April 1964.

2. Davy, Nevil L., *A Mathematical Model of Infantry Combat for SYNTAC Use*, Combat Operations Research Group, Technical Operations, Inc., CORG-M-98, May 1960.

3. Spring, S. G., and S. H. Miller, *FAST-VAL: Relationships Among Casualties, Suppression, and the Performance of Company-Sized Units*, The Rand Corporation, RM-6268-PR, March 1970.

4. Clark, Dorothy K., *Casualties as a Measure of the Loss of Combat Effectiveness of an Infantry Battalion*, Operations Research Office, The Johns Hopkins University, ORO-T-289, August 1954.

5. Weiss, Herbert K., "Stochastic Models for the Duration and Magnitude of a 'Deadly Quarrel,'" *Opns. Res.*, Vol. 11, 1963, pp. 101-121.

6. Horvath, William J., "A Statistical Model for the Duration of Wars and Strikes," *Behavioral Science*, Vol. 13, January 1968, pp. 18-28.

7. Weiss, Herbert K., "Combat Models and Historical Data: The U.S. Civil War," *Opns. Res.*, Vol. 14, 1966, pp. 759-790.

8. Helmbold, Robert L., *Historical Data and Lanchester's Theory of Combat*, Combat Operations Research Group, Technical Operations, Inc., CORG-SP-128, 1 July 1961.

9. Richardson, Lewis F., *Statistics of Deadly Quarrels*, Boxwood Press, Pittsburgh, 1960.

10. Helmbold, Robert L., *Historical Data and Lanchester's Theory of Combat--Part II*, Combat Operations Research Group, Technical Operations, Inc., CORG-SP-190, August 1964.

11. Bodart, Gaston, *Militar-Historisches Kriegs-Lexicon, (1618-1905)*, C. W. Stern, Vienna and Leipzig, 1908.

12. Willard, Daniel, *Lanchester as Force in History: An Analysis of Land Battles of the Years 1618-1905*, Research Analysis Corporation, RAC-TP-74, AD-297375L, November 1962.

13. Schmieman, William A., "The Use of Lanchester-Type Equations in the Analysis of Past Military Engagements," Doctoral Thesis, Georgia Institute of Technology, August 1967.

14. Smith, E. G., and G. N. Donovan, *Trends in Warfare*, British Army Operational Research Group, Memorandum No. F-6, April 1955.

15. Gnedenko, B. V., *The Theory of Probability*, (Chapter XI), 2nd Ed., Chelsea Publishing Co., New York, 1963.

17. Best, Robert J., *A Study of Battle Casualties Among Equivalent Opposing Forces, Korea, September 1950*, Operations Research Office, The Johns Hopkins University, ORO-T-23 (FEC), AD-2885, 14 April 1952.

18. Best, Robert J., *Analysis of Personnel Casualties in the 25th Infantry Division, 26-31 July 1950*, Operations Research Office, The Johns Hopkins University, ORO-T-22 (FEC), AD-171207, 14 April 1952.